Note that for f to be continuous at $x \in X$, we have that
$\forall \varepsilon > 0, \exists V \subseteq X$ open such that $V \subseteq f^{-1}(B_Y(f(x), \varepsilon)$.
Now for each $n \in \mathbb{N}$, let $U_n = \cup\{V|V \subseteq X \text{open}, diam(f(V)) < \frac{1}{n}\}$. Note here that each U_n is an open set. Let $U = \bigcup_{n=1}^{\infty} U_n$.
Let $x \in U$. Then $\exists V_1, V_2, \cdots \subseteq X$ open neighborhoods about x such that $diam(f(V_n)) < \frac{1}{n}$. If $\varepsilon > 0$, then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, and so $\exists V \subseteq X$ open neighborhood about x such that
$diam(f(V)) < \frac{1}{n} < \varepsilon \implies f(V) \subseteq B_Y(f(x), \varepsilon) \implies V \subseteq f^{-1}(B_Y(f(x, \varepsilon))$, so f is continuous at x.
If f is continuous at x, then $\forall n \in \mathbb{N}, \exists V \subseteq X$ open such that
$V = f^{-1}(B_Y(f(x), \frac{1}{n+1}))$, so $x \in U_n \forall n \in \mathbb{N} \implies x \in U$
(as $\text{diam}(f(V)) \leq \frac{1}{n+1} < \frac{1}{n}$), and therefore $U = C_f$, and U (so C_f) is clearly a G_δ set.