Topology Problem 3.16

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(i) $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$:

Proof. Assume that $\mathbb{Q}$ is a G_δ set. Thus we would have

$$\mathbb{Q} = \bigcap_{n\geq 1} G_n$$

Where each G_n is open. Notes that if each open set of X intersects $\mathbb{Q}$, it would necessarily intersect G_n. Thus, G_n is also dense.
Now, take the set

$$U = \{\{q\}^c | q \in \mathbb{Q}\}$$

Since $\mathbb{Q}$ is countable, we note that U is also countable (since it is enumerated by $\mathbb{Q}$). Futhermore, since $\mathbb{Q}$ has no isolated points, we can see that each $\{q\}^c$ is also dense. Thus, by Baire's theorem, we have $\mathbb{Q} \cap \bigcap U \neq \varnothing$. But this is a contradiction because U is made up solely of points not in $\mathbb{Q}$. Therefore $\mathbb{Q}$ cannot be a G_δ set. □

(ii) $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is a G_δ subset of $\mathbb{R}$:

Proof. Note that we can express $\mathbb{I}$ as the intersection of all sets of the form

$$\{q | q \in \mathbb{Q}\}^c$$

Thus since $\mathbb{Q}$ is countable, and $\mathbb{I} = \bigcap_{q\in\mathbb{Q}} \{q\}^c$, $\mathbb{I}$ is an intersection of countably many open sets. Therefore $\mathbb{I}$ is G_δ □

(iii) In a separable Baire space with no isolated points, no countably dense subset is a G_δ set

Proof. Let X be a separable Baire space with no isolated points, and let $Y \subset X$ be countably dense. Since Y is dense, if we have any open set U, then $Y \cap U \neq \varnothing$
Assume Y is a G_δ set. Thus,

$$Y = \bigcap_{n\geq 1} G_n$$

Where G_n is open. Note that since Y is dense, G_n is also dense. Now, since Y is countable,

$$U = \{\{y\}^c | y \in Y\}$$

is a countable intersection of open sets. Furthermore, each $\{y\}^c$ is dense since X has no isolated points.

We now note that $Y \cap \bigcap U \neq \varnothing$ since X is a Baire Space. However, this cannot be the case since clearly $Y \cap \bigcap U = \varnothing$. Therefore we have a contradiction, meaning Y is not a G_δ set. □