Topology Homework 3

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17. Let $f_n : \mathbb{R} \to \mathbb{R}$ continuous. Suppose $f_n \to f$ pointwise on $\mathbb{R}$. Then f is continuous on an uncountably many points of $\mathbb{R}$.

Proof. We are given the following theorem: if X is a Baire space, Y a metric space, $f_n : X \to Y$ continuous, and $f_n \to f$ pointwise, then C_f, the set of points on which f is continuous, must be a dense G_δ set in X.
By the third part of the previous question, we know that in a separable Baire space without isolated points, no countable dense subset is a G_δ set. So, combining these, we can say that f must be continuous on a dense G_δ subset of $\mathbb{R}$, which can't be countable since it's a separable Biare space without any isolated points. Hence, f must be continuous at uncountably many points of $\mathbb{R}$.

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