Topology HW3

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3. Show that X is locally compact if and only if for all compact $C \subset X$ and all open $U \supset C$, there exists an open set V with a compact closure such that $C \subset V \subset \overline{V} \subset U$.

Proof: ($\Longleftarrow$) Assume that for all compact $C \subset X$ and all open $U \supset C$, there exists an open set V with a compact closure such that $C \subset V \subset \overline{V} \subset U$. To see that this space must be locally compact, simply let $C = \{x\}\ \forall x \in X$. If we take any open cover of the set $\{x\}$, taking any one subset produces a finite subcover, so $\{x\}$ is compact. Therefore this is just the definition of locally compact.

($\Longrightarrow$) Now assume that X is locally compact. Take some compact subset C of X. For every $x \in X$, we know that there exists $V_x \subset \overline{V}_x \subset U_x$ where $\overline{V}_x$ is compact. If we take the set of all the V_x, we produce an open cover for C,

$$\{V_x\}_{x \in C}$$

. Since C is compact, there's a finite subset of these that also covers C:

$$\{V_{x_i}\}_{i=1}^{n}$$

Note that for all i, $\overline{V}_{x_i}$ is compact. So if we consider the union of each of these, we get the following:

$$C \subset \bigcup_{i=1}^{n} V_{x_i} \subset \bigcup_{i=1}^{n} \overline{V}_{x_i} \subset \bigcup_{i=1}^{n} U_{x_i}$$

Letting $\bigcup\limits_{i=1}^{n} V_{x_i} = V$, $\bigcup\limits_{i=1}^{n} \overline{V}_{x_i} = \overline{V}$, and $\bigcup\limits_{i=1}^{n} U_{x_i} = U$, we have that

$$C \subset V \subset \overline{V} \subset U$$