MATH 561 - HOMEWORK 3

PATRICK GILLESPIE

8. Show that every locally compact Hausdorff space X is completely regular.

Solution. Let $p \in X$ and $A \subseteq X$ be a closed set such that $p \notin A$. A^c is open and thus, since X is locally compact, we may find an open neighborhood $p \in U_2 \subseteq A^c$ with compact closure $\overline{U}_2$ such that $U_2 \subseteq \overline{U}_2 \subseteq A^c$. Again, but now using U_2 as our open set, we find an open neighborhood $p \in U_1$ with compact closure such that $U_1 \subseteq \overline{U}_1 \subseteq U_2$. Note that $\overline{U}_2 \setminus U_1 = \overline{U}_2 \cap U_1^c$ is closed. Now $\overline{U}_2$ is a compact Hausdorff subspace of X, and so using the fact that every compact Hausdorff space is normal, by Urysohn's lemma, there exists a function $f : \overline{U}_2 \to [0,1]$ such that $f(\overline{U}_2 \setminus U_1) = 0$ and $f(p) = 1$. Define the function $g : U_2^c \to [0,1]$ to be the constant function $g(x) = 0$. Now define $F : X \to [0,1]$ to be f on $\overline{U}_2$ and g on U_2^c. Such a function is well-defined since if $x \in \overline{U}_2 \cap U_2^c$, then $f(x) = 0 = g(x)$. Additionally, since both $\overline{U}_2$ and U_2^c are closed, by the pasting lemma, F is continuous. Hence $F : X \to [0,1]$ is a continuous function such that $F(A) = 0$ and $F(p) = 1$. Therefore X is completely regular.

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