Problem Set 3, Problem 9: A complete metric space without isolated points is uncountable.

Proof. Let (X, d) be a complete metric space without isolated points. Suppose for contradiction that X is countable.

Let $x \in X$. We can make two observations about $X \setminus \{x\}$:

(i) $X \setminus \{x\}$ is open.

For any $y \in X \setminus \{x\}$, we have $x \notin B_{d(x,y)}(y) \subset X \setminus \{x\}$ with $B_{d(x,y)}(y) \neq \emptyset$.

(ii) $X \setminus \{x\}$ is dense is X.

We only need to show that $x \in \overline{X \setminus \{x\}}$. Since x is not an isolated point, we have $(B_{1/n}(x) \setminus \{x\}) \neq \emptyset$. So we can take $x_n \in (B_{1/n}(x) \setminus \{x\})$ for each $n \in \mathbb{N}$ to obtain the sequence $\{x_n\}_{n \in \mathbb{N}}$ in $(X \setminus \{x\})$. Completeness of X implies that x_n converges to x. So x is a limit point of $X \setminus \{x\}$, which implies that $x \in \overline{X \setminus \{x\}}$.

Since complete metric spaces are Baire spaces, $\bigcap_{x \in X}(X \setminus \{x\})$ must be dense in X, but $\bigcap_{x \in X}(X \setminus \{x\}) = \emptyset$, which is a contradiction. $\square$