5 Locally Compact Hausdorff spaces

X(Hausdorff) is loc. compact $\Leftrightarrow$ each $x \in X$ admits a local basis of precompact neighborhoods.

Proof. $\Leftarrow$ Let $x \in X$ be given. Let $\mathcal{B}$ be a local basis at x, then for any open set $U \subset X$ such that $x \in U$, $\exists B \in \mathcal{B}$ such that $U \subset B \subset \overline{B}$. Since, $\overline{B}$ is compact, X is locally compact.

$\Rightarrow$ We show that any open neighborhood U of X contains a precompact one. Let V be a precompact neighborhood of x. Since X is a locally compact Hausdorff space, it is completely regular. Hence the open neighborhood of x , $A = U \cap V$ contains an open neighborhood B s.t. $x \in B \subset \overline{B} \subset A$. Now, B is precompact since $\overline{B}$ is compact(by being a closed subset of a precompact space V). Hence, U contains a precompact neighborhood, B $\square$