Before we get to the problem, recall Proposition 1 and 2:

**Proposition 1.** *$S$ is locally closed in $X \iff S = A \cap F$, where $A$ is open in $X$, and $F$ is closed in $X$ (proof in lecture).*

**Proposition 2.** *Let $X$ locally compact Hausdorff, $S \subset X$ locally closed $\implies S$ is locally compact (proof in problem set 4).*

**Problem Set 4, Problem 17:** Let $X$ be locally compact Hausdorff.

(i) **Closed subsets of $X$ are locally compact.**

Let $S$ be closed in $X$. Note that $S = X \cap S$. Since $X$ is open in $X$, and $S$ is closed, Proposition 1 implies $S$ is locally closed in $X$. Since $X$ is locally compact Hausdorff, Proposition 2 implies $S$ is locally compact.

(ii) **Open subsets of $X$ are locally compact.** (Original proof from class)

Let $S$ be open in $X$. The proof is same as part (i) when we note $S = S \cap X$, since $S$ is open in $X$ and $X$ is closed in $X$.

(ii) **Open subsets of $X$ are locally compact.** (Alternative Proof)

Let $S$ be open in $X$. Let $x \in S$. Since $S$ is open, and $X$ is regular (locally compact Hausdorff spaces are regular), there exists an open neighborhood $V \subset X$ containing $x$ such that $\overline{V} \subset S$ (the overline represents closure in $X$). Local compactness of $X$ implies $x$ has a local basis of precompact neighborhoods. Since $x \in V$, there exists some precompact basis element $B \subset V$. Thus, we have

$$\overline{B} \subset \overline{V} \subset S,$$

so the closure of $B$ in $S$ is $\overline{B}$, which is compact in $X$. Clearly, $\overline{B}$ is compact in $S$ since every open cover of $\overline{B}$ in $S$ is also an open cover of $\overline{B}$ in $X$, which has a finite subcover. Thus, $x \in S$ admits a precompact open neighborhood, which implies $S$ is locally compact.

□