Tariq Islam

Math 561

Problem Set 4

October 1, 2020

Problem 20

$f : X \to Y$ extends continuously to the Alexandrov compactificaion X^* (via $F : X^* \to Y, F(\omega) = y_0$) iff $\lim\limits_{x\to\infty} f(x) = y_0$.

Def : Let X be locally compact Hausdorff and non-compact, Y be a Hausdorff space, $f : X \to Y$ continuous, $y_0 \in Y$. We say $\lim\limits_{x\to\infty} f(x) = y_0$ if for any neighborhood V of y_0 in Y, we may find a compact set $K \subset X$ so that: $x \in X \setminus K \Rightarrow f(x) \in V$.

Proof :
($\Rightarrow$)
Suppose $F : X^* \to Y$ is continuous.

$$F(x) = \begin{cases} f(x) & x \in X \\ y_0 & x = \omega \end{cases}$$

Then for every neighborhood $V \subset Y$ of y_0, $F^{-1}(V)$ is open in X^*.
Now, all the open sets in X^* containing ω are of the form $U \sqcup \{\omega\}$ where $U = X \setminus K$ open and $K \subset X$ compact.
So, for any neighborhood V of y_0 we have $K \subset X$ compact and $x \in X \setminus K \Rightarrow f(x) \in V$.
Therefore, $\lim\limits_{x\to\infty} f(x) = y_0$.

($\Leftarrow$)
Suppose $\lim\limits_{x\to\infty} f(x) = y_0$.
ETS that F is continuous at $\omega \in X^*$.
Let V be any neighborhood of $y_0 \in Y$.
To Show : There exists a neighborhood $U \sqcup \{\omega\} \subset X^*$ of ω such that $\forall x \in U \sqcup \{\omega\}, F(x) \in V$.
Let $K \subset X$ compact so that: $x \in X \setminus K \Rightarrow f(x) \in V$.
Let $U = X \setminus K$.
Claim : $U \sqcup \{\omega\}$ is open in X^*. (Note that U open in X implies $U \sqcup \{\omega\}$ is open in X^*)
Since X Hausdorff and $K \subset X$ compact, K is closed in X.
Therefore, U is open in X.
Hence, F is continuous at $\omega \in X^*$.