Math 561, Topology I, Fall 2020

Set 4 Problem 6

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**6.** If $X$ is first countable and countably compact, then $X$ is regular.

**Proof:**
Let $x \in X$ and $\{U_n\}$ a countable basis for $\mathcal{U}(x)$. Then for each $n$ define

$$V_n = \bigcap_{k=1}^{n} U_k.$$

Notice that $V_{n+1} \subseteq V_n$ and $V_n \subseteq U_n$ for all $n$. Moreover,

$$x \in \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n.$$

Therefore, $\{V_n\}$ is also a countable basis for $\mathcal{U}(x)$. In addition, since $X$ is Hausdorff, we have that

$$y \in \bigcap_{n=1}^{\infty} V_n \iff y \in U \ \forall U \in \mathcal{U}(x) \iff y = x.$$

Which means that $V_n \searrow \{x\}$.

Now, for any $V \in \mathcal{U}(x)$, adding $V$ to the collection $\{\overline{V_n}^c\}$ yields a countable open cover of $X$ because

$$\bigcup_{n=1}^{\infty} \overline{V_n}^c = \left( \bigcap_{n=1}^{\infty} \overline{V_n} \right)^c = \{x\}^c.$$

Furthermore, since $X$ is countably compact, we can find a finite subcover. But notice that $V$ must be in this finite subcover because it's the only set there containing $x$, and we also know that $V_{n+1} \subseteq V_n$, hence $\overline{V_{n+1}} \subseteq \overline{V_n}$, and by induction after taking the complement on both sides, $\overline{V_n}^c \subseteq \overline{V_m}^c$ for any $m > n$. So we can take a finite subcover of the form $\{V, \overline{V_N}^c\}$ for some $N$.

Finally, the fact that this is a cover of $X$ implies that

$$V \bigcup \overline{V_N}^c = X \iff \overline{V_N} \subseteq V.$$

Therefore, $X$ is regular. ■