Let $\mathcal{F}$ be a family of functions $f : X \to Y_f$, X a set and each Y_f a Hausdorff space depending on f. If $\mathcal{F}$ separates points, show that the $\mathcal{F}$ topology on X is Hausdorff.

Proof: The $\mathcal{F}$ topology on X has as a basis the preimages of open sets in each Y_f under the respective functions, so the $\mathcal{F}$ topology on X is the weakest topology which makes each $f \in \mathcal{F}$ continuous.

Let $x, y \in X$ be distinct points. Then since $\mathcal{F}$ separates points, $\exists f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Then these are distinct points in the Hausdorff space Y_f, so there exist disjoint open neighborhoods $U_f, V_f \subseteq Y_f$ such that $f(x) \in U_f, f(y) \in V_f$. Then we get that $f^{-1}(U_f), f^{-1}(V_f)$ are sets open in the $\mathcal{F}$ topology which are disjoint with $x \in f^{-1}(U_f), y \in f^{-1}(V_f)$, so that distinct points are separated by open sets, making the $\mathcal{F}$ topology on X Hausdorff.