TOPOLOGY

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Proof that compact spaces on which there exists a countable family of real continuous functions which separates points are metrizable:

12) Let $\mathcal{F} = \{f_i : X \to \mathbb{R}\}_{i=1}^{\infty}$ be a collection of continuous, real-valued functions on a compact space X. We may, without loss of generality, assume that $|f_i| \le 1$ for each natural number i, since the image of X is compact in $\mathbb{R}$ and therefore bounded. Define the distance function $d : X \times X \to \mathbb{R}$ by

$$d(p,q) = \sum_{i=1}^{\infty} 2^{-i}|f_i(p) - f_i(q)|$$

for any $(p,q) \in X \times X$. That d is a metric on X follows from the triangle inequality for the absolute value function and that $\mathcal{F}$ separates points. Note that

Let $\varepsilon > 0$ and $(p,q) \in X \times X$. Now, there exist an $n \in \mathbb{N}$ so that $2^{-n+1} \le \varepsilon/2$ and $2n$ neighborhoods $U_i, V_i \subset X$ of p and q, respectively, so $x \in U_i$ and $y \in V_i$ implies $|f_i(p) - f_i(x)| < \varepsilon/4$ and $|f_i(q) - f_i(y)| < \varepsilon/4$, since the f_i are continuous. Then, we have, for any $(x,y) \in U_i \times V_i$,

$$|f_i(p) - f_i(q)| - |f_i(x) - f_i(y)| \le |f_i(p) - f_i(x) + (f_i(y) - f_i(q))| \le |f_i(p) - f_i(x)| + |f_i(q) - f_i(y)| < \frac{\varepsilon}{2}$$

for each $i \le n$ and

$$|f_i(p) - f_i(q)| - |f_i(x) - f_i(y)| \le 2$$

for any $i > n$. Then, if we set $U = \bigcap_{i=1}^{n} U_i \times V_i$, then U is a neighborhood of $(p,q) \in X \times X$ and if $(x,y) \in U$, then

$$|d(p,q) - d(x,y)| = \left|\sum_{i=1}^{n} 2^{-i}(|f_i(p) - f_i(q)| - |f_i(x) - f_i(y)|) + \sum_{i=n+1}^{\infty} 2^{-i}(|f_i(p) - f_i(q)| - |f_i(x) - f_i(y)|)\right|$$

$$\le \sum_{i=1}^{n} 2^{-i}\frac{\varepsilon}{2} + \sum_{i=n+1}^{\infty} 2^{-i} \cdot 2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Then, $d : X \times X \to \mathbb{R}$ is continuous on $X \times X$ in the product topology, so certainly $B_d(p,r) = \pi\left(d|_{\{p\}\times X}^{-1}([0,r))\right)$ is open in the product topology as well, since projection maps are open. Thus, $\mathcal{T}_d \subset \mathcal{T}$. Since metric topologies are always Hausdorff, and $\mathcal{T}$ is compact, then we automatically have that $\mathcal{T}_d = \mathcal{T}$.

Alternatively (although less desirable), here is a direct proof that $\mathcal{T} \subset \mathcal{T}_d$:

Now we show $\mathcal{T} \subset \mathcal{T}_d$. Let $U \in \mathcal{T}$ and $x \in U$. Consider $\{\text{int}(X \setminus B_d(x,r))\}_{r>0}$. This is an open cover of $X \setminus U$, which, being a closed subspace of a compact space, is compact. So, there exists an $r > 0$ so $\text{int}(X \setminus B_d(x)) \supset X \setminus U$, implying $B_d(x,r) \subset U$.