Math 561, Topology I, Fall 2020

Set 5 Problem 4

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**4.** Let $(Y, e_Y)$ be any compactification of $X$ and $(\beta X, e_\beta)$ the Stone-Cech compactification of $X$, there exists a continuous surjective closed map $F : \beta X \to Y$ which satisfies $F \circ e_\beta = e_Y$.

**Proof:**

We know that $e_Y : X \to Y$ is continuous and $Y$ is compact. Thus by the universal extension property of $\beta X$, there is a unique continuous map $F : \beta X \to Y$ such that $F \circ e_\beta = e_Y$. Moreover, since $Y$ is a compactification of $X$ we must have that $\overline{e_Y(X)} = Y$, so $\overline{F\left(e_\beta(X)\right)} = Y$. But $F$ is continuous and $\beta X$ is also a compactification of $X$, hence

$$Y = \overline{F\left(e_\beta(X)\right)} = F\left(\overline{e_\beta(X)}\right) = F\left(\beta X\right).$$

That is, $F$ is surjective.

Finally, notice that any closed subset of $\beta X$ is compact and by continuity of $F$ its image is also compact and therefore closed in $Y$. Hence, $F$ is also a closed map. ■