Topology Problem 5.5

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The following two theorems are from *Munkers*

Theorem 23.4: Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Theorem 38.4: Let X be a completely regular space; let Y be a compactification of X satisfying the extension property of Theorem 38.2. Given any continuous map $f : X \to C$ of X into a compact Hausdorff space C, the map f extends uniquely to a continuous map $g : Y \to C$.

Theorem. *X is connected if and only if βX is connected.*

Proof. ($\Longrightarrow$) Let X be connected. Note that since $X \subset \beta X \subset \bar{X}$ (in fact, $\beta X = \bar{X}$), then (by Theorem 23.4 in Munkres) it follows that βX must be connected.

($\Longleftarrow$) Now following the hint from Munkres (by means of contraposition), let X be disconnected and let $A \cup B$ be a separation of X. Now we define the continuous function $f : X \to \{0, 1\}$ such that $f(A) = 0$ and $f(B) = 1$. Note, that by Theorem 38.4 in Munkres, we can extend f such that $f^* : \beta X \to \{0, 1\}$ is continuous. Note that the preimage of $\{0\} \cup \{1\}$ is equal to βX, but since $f^{-1}\{0\} \cap f^{-1}\{1\} = \varnothing$, it follows that βX is disconnected.

$\square$