We will first prove that the product of countably many second countable spaces is second countable.

Let I be a countable index and let X_i be a second countable topological space with countable basis β_i for $i \in I$. Then we know that an arbitrary basis of the product topology of X is given by

$$B = \pi_{n_1}^{-1}(B_{n_1}) \cap \pi_{n_2}^{-1}(B_{n_2}) \cap \cdots \cap \pi_{n_m}^{-1}(B_{n_m}),$$

where $B_{n_i} \in \beta_{n_i}$ for $i \in \{1, 2, \ldots, m\}$. Since β_n is countable for each n, the set of all such B's must be countable. Thus, X has a countable basis. We conclude that a countable producy of second countable spaces is second countable.

Now, for $f \in \mathcal{F}$ each $I_f = [\inf f, \sup f]$ is second countable. thus, $P = \prod_{f\in\mathcal{F}}^{\infty} I_f$ is a countable product of second countable spaces. So P is second countable. Since $\hat{X}$ is a subspace of P, it is also second countable. $\square$