# Probelm 5: ***Sorgenfrey line***, denoted $\mathbb{R}_\ell$

**(i)** The collection of subsets of $\mathbb{R}$, $\mathcal{B} = \{[a, b) : a < b\}$ (left-closed intervals) define a basis of a topology on $\mathbb{R}$.

We will show that $\mathcal{B} = \{[a, b) : a < b\}$ is a basis on $\mathbb{R}$.

1. For each $x \in \mathbb{R}$, we have $x \in [x, x + 1) \in \mathcal{B}$. Thus, there exists $B \in \mathcal{B}$ such that $x \in B$ for each $x \in \mathbb{R}$.

2. Let $x \in B_1 \cap B_2$, where $x \in \mathbb{R}$, $B_1 = [a_1, b_1) \in \mathcal{B}$, and $B_2 = [a_2, b_2) \in \mathcal{B}$. Since $B_1 \cap B_2$ is non-empty, we must have

$$\max(a_1, a_2) \leq x < \min(b_1, b_2).$$

Set $B_3 = [\max(a_1, a_2), \min(b_1, b_2)) \in \mathcal{B}$. Since $B_3 = B_1 \cap B_2$, we conclude that for each $x \in \mathbb{R}$ such that $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

**(ii)** The topology is Hausdorff, and is finer than the usual topology on $\mathbb{R}$.

To show that the topology is Hausdorff, let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$. Assume, without loss of generality, that $x_1 < x_2$. Let $B_1 = [x_1, x_2)$, and $B_2 = [x_2, x_2 + 1)$. Note that $B_1, B_2 \in \mathcal{B}$, so they are both open. Additionally,

$$x_1 \in B_1, \qquad x_2 \in B_2, \qquad B_1 \cap B_2 = \emptyset. \tag{1}$$

This proves that the topology generated by $\mathcal{B}$ is Hausdorff.

To show that $\mathbb{R}_\ell$ is finer than the usual topology, recall that the set of open intervals in $\mathbb{R}$ is a basis of the standard topology. Let $x \in \mathbb{R}$, and let $x \in (a, b)$. Since $a < x < b$,

$$x \in [x, b) \subset (a, b). \tag{2}$$

Note that $[x, b) \in \mathcal{B}$, and $(a, b)$ is an arbitrary basis element of the usual topology that contains $x$. By lemma 13.3 from Munkres, we conclude that the topology generated by $\mathcal{B}$ is finer than the standard topology.

**(iii)** $\lim x_n = a$ in $\mathbb{R}_\ell$ iff $x_n \to a_+$ in the usual topology.

In order to prove that $x_n \to a$ in any topological space, it is sufficient to show that for each basis element, $B$, that contains $a$, there exists some positive integer $N$ such that $x_n \in B$ for each $n \geq N$. This is because for each open set $U$ that contains $a$, there exists a basis element $B$ such that $a \in B \subset U$. Also, since $\mathbb{R}_\ell$ is Hausdorff, it makes sense to say that the limit "equals" an element.

( $\Longleftarrow$ ) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $x_n \to a_+$ in the usual topology. Let $B = [b, c)$, be an element of $\mathcal{B}$ that contains $a$.

Consider $(b-1, c)$, which is open in the usual topology. Since $x_n \to a_+$ in the usual topology, there exists some positive integer $N_1$ such that $x_n \in (b-1, c)$ for each $n \geq N_1$. Because the limit is right-sided, there exists some positive integer $N_2$ such that $x_n \notin (b-1, a)$ for each $n \geq N_2$. Thus, for $N := \max(N_1, N_2)$,

$$x_n \in (b-1, c) - (b-1, a) = [a, c) \subset [b, c) = B \tag{3}$$

for each $n \geq N$. This implies $\lim x_n = a$ in $\mathbb{R}_\ell$.

( $\Longrightarrow$ ) Suppose $\lim x_n = a$ in $\mathbb{R}_\ell$. Let $B = (b, c)$ be a basis of the standard topology on $\mathbb{R}$ that contains $a$. Since $\mathbb{R}_\ell$ is finer than the usual topology, $B$ is also open in $\mathbb{R}_\ell$. So, there exists some positive natural number $N$ such that for each $n \geq N$, we have $x_n \in B$. Thus, $x_n$ converges to $a$ in the usual topology.

To show that $(x_n)$ converges from the right, suppose not. Then there must be infinitely many $n$ such that $x_n \in (-\infty, a)$. On the other hand, $a \in [a, a+1) \in \mathcal{B}$. Thus, $x_n \in [a, a+1)$ for all sufficiently large $n$. Thus, there must exist some $n$ such that $x_n \in [a, a+1)$ and $x_n \in (-\infty, a)$. But,

$$(-\infty, a) \cap [a, a+1) = \emptyset, \tag{4}$$

which is a contradiction. We conclude that $x_n \to a_+$ in the usual topology.