MATH 561 - ASCOLI ARZELA

PATRICK GILLESPIE

Problem 10. A sequence of polynomials of degree $\leq k$, uniformly bounded in a compact interval, is equicontinuous on this interval.

Solution. It suffices to show that if a sequence of polynomials of degree less than or equal to k is uniformly bounded on $[0,1]$, then it is equicontinuous on $[0,1]$. This is because, if a sequence of polynomials $\{f_n\}$ is uniformly bounded on $[a,b]$, then the sequence of polynomials $\{g_n\}$ defined by $g_n(x) = f_n((b-a)x+a)$ is uniformly bounded on $[0,1]$ and is equicontinuous on $[0,1]$ if and only if $\{f_n\}$ is equicontinuous on $[a,b]$.

Note that the set of polynomials of degree less than or equal to k form a finite dimensional vector space V. Given a polynomial $f \in V$, let a_i be the coefficient of x^i in f. Both $\max_{x\in[0,1]}\{|f(x)|\}$ and $\max_{0\leq i\leq k}\{|a_i|\}$ define norms on V. Since all norms on a finite dimensional vector space are equivalent, this implies that if a sequence of polynomials $\{f_n\}$ is uniformly bounded on a $[0,1]$, the supremum of the coefficients of f_n over all $n \in \mathbb{N}$ is finite.

So suppose $\{f_n\}$ is a sequence of polynomials of degree less than or equal to k, uniformly bounded on $[0,1]$. Set $M = \sup\{|a_{i,n}| : i \in \{0,1,\dots,k\}, n \in \mathbb{N}\}$ where $a_{i,n}$ is the coefficient of x^i in f_n. Define the polynomials g_n by $g_n(x) = \frac{1}{M}f_n(x)$. We show that $\{g_n\} = \{\frac{1}{M}f_n\}$ is equicontinuous on $[0,1]$. It is easy to check that this implies that $\{f_n\}$ is equicontinuous on $[0,1]$.

Write $g_n = c_{k,n}x^k + c_{k-1,n}x^{k-1} + \cdots + c_{0,n}$. Then

$$g_n' = kc_{k,n}x^{k-1} + (k-1)c_{k-1,n}x^{k-2} + \cdots + c_{1,n}.$$

By the definition of g_n, $|c_{i,n}| \leq 1$ for all $i \in \{0,1,\dots,k\}$ and $n \in \mathbb{N}$. Hence for all $x \in [-2,2]$, we have that $|g_n'(x)| \leq \sum_{i=1}^{k} i \cdot 2^{i-1}$ for all $n \in \mathbb{N}$. Therefore $\{g_n'\}$ is uniformly bounded on $[-2,2]$, say, by N. We now show that $\{g_n\}$ is equicontinuous on $[0,1]$. Let $x_0 \in [0,1]$. Let $\epsilon > 0$. Set $\delta = \min\{1, \frac{\epsilon}{N}\}$. Suppose $|x - x_0| < \delta$ and that there exists $n \in \mathbb{N}$ such that $|g_n(x) - g_n(x_0)| \geq \epsilon$. Then by the mean value theorem, there exists a point c in the open interval between x and x_0 such that $g_n'(c) = \frac{g_n(x)-g_n(x_0)}{x-x_0} > \frac{\epsilon}{\delta} \geq N$, contradicting the fact that $\{g_n'\}$ is uniformly bounded on $[-2,2]$ by N. Hence $|x - x_0| < \delta$ implies that $|g_n(x) - g_n(x_0)| < \epsilon$ for all $n \in \mathbb{N}$. Thus $\{g_n\}$ is equicontinuous on $[0,1]$. □

Date: October 24, 2020.