Problem Set 6, Problem 11: Let (f_n) be an equicontinuous and pointwise bounded sequence of functions in $C_E(X)$ (E Banach and finite dimensional, X compact metric.) If every uniformly convergent subsequence has the same limit $f \in C_E(X)$, then f_n converges to f uniformly on X.

Proof. We want to apply Arzela Ascoli Theorem to $\mathcal{F} := \{f_n\}$. Equicontinuity is given, so we must show pointwise compactness. Fix some $x \in X$ and consider the set $\overline{\{f_n(x)\}} \subset E$. Since $\mathcal{F}$is pointwise bounded, the set $\{f_n(x)\}$ is bounded. Meaning $\overline{\{f_n(x)\}}$ is bounded and closed. Since finite dimensional Banach spaces are Heine-Borel, this implies that $\overline{\{f_n(x)\}}$ is compact, meaning $\{f_n(x)\}$ is precompact. This satisfies the precompactness condition for applying Arzela-Ascoli theorem to F.

By Arzela Ascoli Theorem, $\overline{\mathcal{F}}$ is compact. Now, suppose for contradiction that f_n does not converge to f. So there exists some $\varepsilon > 0$ such that

$$\|f_n - f\| \geq \varepsilon \tag{$*$}$$

for infinitely many n's Let $f_{n_1}, f_{n_2}, \ldots$ be a subsequence of (f_n) that satisfy $(*)$. By sequential compactness of $\overline{\mathcal{F}}$, there exists a futher subsequence, $f_{n_{m_1}}, f_{n_{m_2}}, \ldots,$ that converges in $\overline{\mathcal{F}}$. Since convergence in $\overline{\mathcal{F}}$ is the same as uniform convergence, we know that $f_{n_{m_i}} \to f$. But this is impossible since $f_{n_{m_i}}$ satisfy $(*)$, a contradiction. □