Math 561, Topology

Set # 6, Arzela-Ascoli Notes

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Problem 3: If each $f_n : X \to E$ (X metric, E Banach) is uniformly continuous on X and $f_n \to f$ uniformly on X, then f is uniformly continuous on X.

Solution:

Let $\epsilon > 0$. We want $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$ we have $\|f(x) - f(y)\| < \epsilon$. First rewrite the left side of that inequality as

$$\|f(x) - f(y)\| = \|f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)\|.$$

Then, choose $n \in \mathbb{N}$ such that for any $x \in X$ we have

$$\|f(x) - f_n(x)\| < \frac{\epsilon}{3}.$$

This is possible by uniform convergence. Now take $\delta > 0$ such that for any $x, y \in X$ such that $d(x, y) < \delta$ we have that

$$\|f_n(x) - f_n(y)\| < \frac{\epsilon}{3}.$$

Which is possible by uniform continuity of each f_n.

Hence for any $x, y \in X$ such that $d(x, y) < \delta$ we must have

$$\begin{aligned}\|f(x) - f(y)\| &\leq \|f(x) - f_n(x)\| + \|f_n(x) - f_n(y)\| + \|f_n(y) - f(y)\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore f is uniformly continuous on X. ■

Problem 4: There is no sequence of polynomials converging either to $1/x$ or to $\sin(1/x)$ uniformly on the open interval $(0, 1)$.

Solution:

Using the previous problem, take X as the closed interval $[0, 1]$ and notice that any polynomial is uniformly continuous on X because polynomials are continuous and X is compact. However, neither $1/x$ or $\sin(1/x)$ are uniformly continuous on X since there is no way to define them at 0 in such a way that they are still continuous. Therefore, we cannot have a sequence of polynomials converging uniformly to these functions, for if this was possible, we could conclude from the previous problem that these functions are uniformly continuous. ■