6 Convergence of Monotone functions on a Compact set

8. If a sequence of real-valued monotone functions (with domain $\mathbb{R}$) converges pointwise to a continuous function on an interval $I \subset \mathbb{R}$, then the convergence is uniform on each compact subset of I

Proof. We shall prove this for a sequence of monotone increasing functions, and show how it can be extended to monotone decreasing functions.
Let f_n be a sequence of monotone functions (on $\mathbb{R}$) such that on $I \in \mathbb{R}$, $f_n \to f$ pointwise (where f is continuous). Let $\epsilon > 0$ be given.
RTS: $\exists N \in \mathbb{N}$ such that for $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon \text{ for } x \in A \subset I$$

where A is compact.
Since $A \subset R$ is compact, by Heine-Borel, it is closed and bounded. Hence it is contained in some closed interval $C \subset I$. WLG, let $C = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < b$
Furthermore, f is uniformly continuous on $[a, b]$ (continuous functions on compact sets). Hence

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } \forall x, y \in [a, b],$$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Fix such δ. Then by the Archimedean Property, $\delta > \frac{b-a}{M}$ for some integer M.

Next, we generate M subdivisions of $[a, b]$: $P_1, \cdots, P_M$ where $P_i = [x_i, x_{i+1}]$ for $i = 0, \cdots, M-1$ and $(a, b) = (x_0, x_M)$. By the uniform continuity of f(as shown above),

$$|f(x) - f(y)| < \frac{\epsilon}{2} \text{ for } x, y \in P_i \text{ where } i = 1, \cdots, M$$

Since $f_n \to f$ pointwise, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|f_n(x_i) - f(x_i)| < \frac{\epsilon}{2} \text{ for } i = 0, \cdots, M \tag{1}$$

Now, to see that $|f_n(x) - f(x)| < \epsilon$ for $n \geq N$ and $x \in [a, b]$, we start by fixing $n \geq N$.
Let $x \in [a, b]$ be given. It follows that $x \in P_i$ for $i = 0, \cdots, M-1$
Since f_n is monotone increasing,

$$f_n(x_i) \leq f_n(x) \leq f_n(x_{i+1}) \tag{2}$$

Furthermore, from (1), $f(x_i) - \frac{\epsilon}{2} < f_n(x_i)$ and similarly, $f_n(x_{i+1}) \leq f(x_{i+1}) + \frac{\epsilon}{2}$.

Therefore,

$$f(x_i) - \frac{\epsilon}{2} \leq f_n(x) \leq f(x_{i+1}) + \frac{\epsilon}{2} \tag{3}$$

By the uniform continuity of f on P_i,

$$|f(x) - f(x_i)| < \frac{\epsilon}{2} \Rightarrow f(x_i) > f(x) - \frac{\epsilon}{2} \tag{4}$$

and similarly,

$$|f(x) - f(x_{x+1})| < \frac{\epsilon}{2} \Rightarrow f(x_{i+1}) < f(x) + \frac{\epsilon}{2} \tag{5}$$

Plugging these results into (3) yields,

$$f(x) - \epsilon < f(x_i) - \frac{\epsilon}{2} \leq f_n(x) \leq f(x_{i+1}) + \frac{\epsilon}{2} < f(x) + \epsilon$$

Therefore, $|f_n(x) - f(x)| < \epsilon$ □

Remark: For the case were f_n was a sequence of monotone decreasing functions, the proof will be almost identical. We would only need to reverse the inequalities in (2) and replace i with $i+1$ and vice-versa in (4) and (5) respectively.