Topology Homework 7.10

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10 Let X be any space, (Y, d) metric. For each $f : X \to Y, \epsilon > 0, C \subset X$ compact, consider the subset of $\mathcal{F}(X, Y)$:

$$B_C(f, \epsilon) = \{g : X \to Y; d(f(x), g(x)) < \epsilon, \forall x \in C\}$$

Show these sets form the basis of a topology in $\mathcal{F}(X, Y)$.

Proof. We must show that these sets, $B_C(f, \epsilon)$, form a basis for a topology. In other words, we show every $g \in \mathcal{F}$ belongs to a set of this type, and that every one of these are contained in an intersection of two others.
So, let $g \in \mathcal{F}$. We choose $\epsilon = \max\{d(f(x), g(x)), x \in C\}$. Then obviously $g \in B_C(f, \epsilon)$.
To see that every intersection of two basis elements contains a third, consider two sets $B_{C_1}(f, \epsilon_1)$ and $B_{C_2}(f, \epsilon_2)$ for $\epsilon_1, \epsilon_2 > 0$. Then if we take

$$\delta = \min\{(\epsilon_1 - \sup\{d(f(x), g(X)), \forall x \in C_1\}), (\epsilon_2 - \sup\{d(f(x), g(X)), \forall x \in C_2)\}$$

and let $C = C_1 \cup C_2$, we can conclude that the set $B_C(f, \delta) \subset B_{C_1}(f, \epsilon_1) \cap B_{C_2}(f, \epsilon_2)$. Thus, these sets form a basis of the compact convergence topology.

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