MATH 561 - PROBLEM SET 7

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Problem 18. If X is any space and (M, d) is a complete metric space, let $\mathcal{F} = \{f_n\}_{n\geq 1} \subset C(X; M)$ be a countable equicontinuous set. If $f_n(x)$ converges for all x in a dense subset $D \subset X$, show that f_n converges u.o.c in X.

Solution. Given a compact set $K \subset X$, we show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f_m(x)) < \epsilon$ for all $m, n \geq N$ and $x \in K$. Since (M, d) is complete, this will show that f_n converges uniformly on K. So let $\epsilon > 0$. By the equicontinuity of $\mathcal{F}$, for every $x \in K$, there exists an open neighborhood U_x of x such that $d(f_n(x), f_n(y)) < \frac{\epsilon}{5}$ for all $y \in U_x$ and $n \in \mathbb{N}$. The union over all $x \in K$ of such neighborhoods, $\cup_{x\in K} U_x$, is an open cover of K. Hence there exists a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_k}\}$. Note that since D is dense in X, every U_{x_i} contains an element $d_i \in D$. Also, $f_n(d_i)$ converges for all $i \in \{1, 2, \ldots, k\}$. Hence for each i there exists $N_i \in \mathbb{N}$ such that $d(f_n(d_i), f_m(d_i)) < \frac{\epsilon}{5}$ for all $m, n \geq N_i$. Set $N = \max\{N_i \mid 1 \leq i \leq k\}$. Now let $x \in K$. Then $x \in U_{x_i}$ for some $i \in \{1, 2, \ldots, k\}$. Since x and d_i are both elements of U_{x_i}, by the triangle inequailty, $d(f_n(x), f_n(d_i) < \frac{2\epsilon}{5}$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} d(f_n(x), f_m(x)) &\leq d(f_n(x), f_n(d_i)) + d(f_n(d_i), f_m(d_i)) + d(f_m(d_i), f_m(x)) \\ &\leq \frac{2\epsilon}{5} + \frac{\epsilon}{5} + \frac{2\epsilon}{5} = \epsilon \end{aligned}$$

for all $m, n \geq N$. This completes the proof. $\square$

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