7 Countable Product of Sequentially Compact Spaces

9a. If $Y_1, Y_2, \cdots$ are sequentially compact, then $Y = \Pi_{n\geq 1} Y_n$ is sequentially compact.

Proof. Let $S_k = \Pi_{n\geq 1}(y_{nk})_{k\in\mathbb{N}}$ be a sequence in Y. We perform a variant of the diagonal argument as follows: Fix, the first term of the sequence, and then , select a convergent subsequence for the first component(Y_1). Thin out the rest of the sequence so that the terms for each component, the remaining terms are corresponding terms of the convergent sequence.
After the first iteration,
$S_k^1 = \Pi_{n\geq 1}[y_{n1}, (y_{nk}^1)_{k>1}]$ where $\{y_{1k}^1\}_{k>1} \subset \{y_{1k}\}_{k>1}$ that converges to some $a_1 \in Y_1$. We observe that the first component of the sequence converges, and also, the first term for all proceeding components is fixed.
Next, we fix the second term of this new sequence and choosing a convergent subsequent for the second component, we repeat the "thinning out" process.
After the second iteration,
$S_k^2 = \Pi_{n\geq 1}[y_{n1}, y_{n2}^1, (y_{nk}^2)_{k>2}]$ where $\{y_{2k}^2\}_{k>1} \subset \{y_{2k}^1\}_{k>1}$ that converges to some $a_2 \in Y_2$.
At this point, we observe that the first and second components of the sequence converges, and we have simultaneously fixed the second term for all proceeding components.

Continuing in this fashion, at the mth iteration, the first m components of the sequence will be convergent. Therefore, $y_{n1}, y_{n2}^1, y_{n3}^2, \cdots$ is a subsequence of our original sequence that converges in the product topology; and hence our proof is complete. □

9b. If N is countable and Y is sequentially compact, $\mathcal{F}_p(N, Y)$ is sequentially compact.

Proof. $\mathcal{F}_p(N, Y) = \{f(p) : f \in \mathcal{F}\}$ for $p \in N$. Since N is countable, $\{f(N)\}$ is countable $\forall f \in \mathcal{F}$
Let $(f_n(p))_{n\geq 1}$ be a sequence in $\mathcal{F}_p(N, Y)$. $\{f_n(p)\}_{n\geq 1} \subset Y$ and therefore, it has a convergent subsequence. Thus , our proof is complete. □