Topology Problem Set 7

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1. Let X be a σ-compact space and (Y, d) a metric space. Show that Y^X in the u.o.c. topology is metrizable, and with a complete metric if Y is complete.

Proof. Let $X_i \subset X$ for $i \in \mathbb{N}$ be compact subspaces so $\bigcup_i X_i = X$. Consider $Y_i = Y^{X_i}$, the collection of functions from X_i to Y in the uniform topology given by the metric

$$d_i(f,g) := \sup_{x \in X_i} \min\{d(f(x), g(x)), 1\} = \sup_{x \in X_i} d^{\mathrm{bd}}(f(x), g(x)), \qquad \forall f, g : X_i \to Y$$

where $d^{\mathrm{bd}} = \min\{d, 1\}$ is the so-called standard bounded metric on Y, which induces the same topology as d. It was shown in Theorem 43.5 of Munkres that this metric d_i on Y_i is complete if Y is. Thus, the product space

$$P := \prod_{i=1}^{\infty} Y_i$$

with the product topology is metrizable, and completely metrizable if Y is—Patrick showed this in Exercise 2 of our section on complete metrizability. Specifically, one metric is

$$d_P((f_i)_{i=1}^{\infty}, (g_i)_{i=1}^{\infty}) := \sum_{i=1}^{\infty} \frac{d_i(f_i, g_i)}{2^i(1 + d_i(f_i, g_i))}, \qquad \forall f_i, g_i \in Y_i$$

though all we will need is that this metric d_P induces the product topology on P and is complete if Y is complete.

Define $\varphi : Y^X \to P$ by, for any function $f : X \to Y$, $\varphi(f) = (f|_{X_i})_{i=1}^{\infty}$. That is, the i^{th} component of $\varphi(f)$ is the function f restricted to X_i. This function φ is clearly injective, for if two functions f and g are so their restrictions to every set X_i are equal, then $f = g$ since the X_is cover X. Considered as a map from Y^X to $\varphi(Y^X) \subset P$, then, φ is bijective. We seek also to show that this restriction of the range makes φ a homeomorphism.

We start by showing φ is continuous. We'll argue with basis elements: let $B = \prod_i U_i = U_1 \times U_2 \times \cdots \subset P$ be a basis element in P, so there is an $n \in \mathbb{N}$ so that $i > n$ implies $U_i = Y_i$. Let $f \in \varphi^{-1}(B)$, and let

$$B_i := B_{d_i}(f, \varepsilon_i) = \{g : X_i \to Y : d_i(f,g) < \varepsilon_i\}$$

be a basic open set in Y_i for $1 \le i \le n$ so that $f \in B_i \subset U_i$. Then, since the finite union

$$C := \bigcup_{i=1}^{n} X_i$$

is compact in X, then, setting $\delta = \frac{1}{2}\min\{1/2, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\} > 0$, the u.o.c. basic open set

$$U = B_C(f, \delta) = \left\{ g : X \to Y : \sup_{x \in C} d(g(x), f(x)) < \delta \right\}$$

about f is such that $U \subset \varphi^{-1}(B)$, since if $g \in U$ then $\sup_{x\in X_i} d(f(x), g(x)) < \frac{1}{2}, \varepsilon_i$ so $d_i(f,g) = \sup_{x\in X_i} d(f(x), g(x))$ and $d_i(f,g) < \varepsilon_i$, so $\varphi(g) \in B_i \subset U_i$ for each $i \leq n$. This shows φ is continuous.

Now we show φ is open *as a map from Y^X to $\varphi(Y^X)$*. Again we argue from basis elements: let $f \in Y^X$, $C \subset X$ be compact, and $\varepsilon > 0$. Since X is locally compact, we may assume that the collection of interiors $\{\text{int}(X_i)\}_{i\in\mathbb{N}}$ also covers X. Thus, this collection also covers C, so there is a positive integer n so that the interiors of the sets X_i, $i \leq n$, cover C. Thus, if we choose a $\delta > 0$ so $\delta < 1$ and $\delta < \varepsilon$, then if $d_i(f|_{X_i}, g|_{X_i}) < \delta$ for each $i \leq n$, then $\sup_{x\in X} d(f(x), g(x)) < \delta < \varepsilon$. Thus, the basic open set

$$\varphi\left(Y^X\right) \cap \left(B_{d_1}(f|_{X_1}, \delta) \times B_{d_2}(f|_{X_2}, \delta) \times \cdots \times B_{d_n}(f|_{X_n}, \delta) \times Y_{n+1} \times Y_{n+2} \times \cdots\right)$$

in $\varphi(Y^X)$ is contained in the image of $B_C(f,\varepsilon)$, so φ is a homeomorphism from Y^X to its image. Furthermore, $\varphi(X)$ is closed in P, because for any $(g_i)_{i=1}^{\infty} \notin \varphi(X)$, there must exist X_i and X_j so $X_i \cap X_j \neq \emptyset$ and g_i and g_j do not agree on this intersection. This means that

$$\sup_{x\in X_i\cap X_j} d^{\text{bd}}(g_i(x), g_j(x)) := \varepsilon > 0.$$

Thus, if $(f_i)_{i=1}^{\infty}$ is another element of P for which $f_i \in B_{d_i}(g_i, \varepsilon/4)$ and $f_j \in B_{d_j}(g_j, \varepsilon/4)$, then, since there exists an $x \in X_i \cap X_j$ so

$$d^{\text{bd}}(g_i(x), g_j(x)) > \frac{3\varepsilon}{4},$$

then for that x, we have that

$$d^{\text{bd}}(f_i(x), f_j(x)) \geq d^{\text{bd}}(g_i(x), g_j(x)) - d^{\text{bd}}(g_i(x), f_i(x)) - d^{\text{bd}}(g_j(x), f_j(x)) > \frac{3\varepsilon}{4} - \frac{\varepsilon}{2} = \frac{\varepsilon}{4},$$

so

$$\sup_{x\in X_i\cap X_j} d^{\text{bd}}(f_i(x), f_j(x)) \geq \frac{\varepsilon}{4}$$

and (f_i) could not be in $\varphi(Y^X)$ either. Thus, the open set $U_1 \times U_2 \times \cdots$ in P where $U_i = B_{d_i}(g_i, \varepsilon/4)$ and $U_j = B_{d_j}(g_j, \varepsilon/4)$ and $U_k = Y_k$ for $k \neq i, j$ is a basic open set about (g_i) which does not intersect $\varphi(Y^X)$ and the image of φ must be closed in P. This guarantees that the image is completely metrizable if Y is complete, since it would be closed subset of the complete metric space P. $\square$

17. Denote by $\mathcal{H}$ the collection of all homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. This, being a subset of $\mathbb{R}^{\mathbb{R}}$, may inherit both the topology of pointwise convergence $\mathcal{T}_p$ or the topology of uniform convergence on compact sets $\mathcal{T}_{uc}$. We already know that the former is a coarser topology than the latter (i.e., $\mathcal{T}_p \subset \mathcal{T}_{uc}$), so we seek to show that the converse is also true on $\mathcal{H}$. For this, it suffices to show that for any $f \in \mathcal{H}$, compact $C \subset \mathbb{R}$, and $\varepsilon > 0$, the basic open set

$$B_C(f, \varepsilon) = \left\{ g \in \mathcal{H} : \sup_{x \in C} |f(x) - g(x)| < \varepsilon \right\}$$

contains a neighborhood $U \in \mathcal{T}_p$ of f. Additionally, since any compact subset of the real line is contained in a compact interval $I \supset C$, so then

$$B_C(f, \varepsilon) \supset B_I(f, \varepsilon),$$

and we may without loss of generality assume C is an interval.

Note also that for any $f \in \mathcal{H}$, it is easily seen that f is either strictly increasing or strictly decreasing—take the connected subset $H = \{(x_1, x_2) : x_1 < x_2\}$ of $\mathbb{R}^2$ and note that the map $D : H \to \mathbb{R}$ given by $D(x_1, x_2) = f(x_2) - f(x_1)$ is never zero, since f is injective, so $D(H)$ is either positive (so f is increasing) or negative (so f is decreasing) since it is clearly continuous.

Proof. Let $f \in \mathcal{H}$, $C = [a, b]$ be a compact interval, and $\varepsilon > 0$. Since f is continuous and injective, it maps C onto some other compact interval $[c, d]$ for some $c < d$, with $f(a) = c$ and $f(b) = d$ if f is increasing or with $f(a) = d$ and $f(b) = c$ if f is decreasing. Without loss of generality, suppose f is increasing; the proof remains largely unchanged, just with certain inequalities reversed. Choose a partition

$$f(a) = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = f(b)$$

of the image $f(C)$ so that the $\max_{1 \le i \le n}(y_i - y_{i-1}) < \frac{\varepsilon}{2}$. Denote also by x_i the unique inverse $f^{-1}(y_i)$ for $0 \le i \le n$. Choose $\delta = \frac{1}{2} \min\{\varepsilon, y_n - y_0\}$ and consider the $\mathcal{T}_p$-open set

$$U = \{g \in \mathcal{H} : \forall i \in \{0, 1, \cdots, n\},\ |g(x_i) - y_i| < \delta\}.$$

If $g \in U$, then since $|g(x_0) - y_0|$ and $|g(x_n) - y_n|$ are less that $\frac{y_n - y_0}{2}$, then $g(x_0) < \frac{y_n + y_0}{2} < g(x_n)$, so g is also strictly increasing; i.e., $g(x_0) < g(x_1) < \cdots < g(x_n)$. We already know that $g(x_i)$ is within ε of $y_i = f(x_i)$ for any $i = 0, 1, 2, \cdots, n$, so let $x \in (x_{i-1}, x_i) \subset C$ for some such $i \ge 1$. Then,

$$g(x_{i-1}) < g(x) < g(x_i),$$

and since $-f$ is decreasing, then

$$g(x_{i-1}) - y_i < g(x_{i-1}) - f(x) < g(x) - f(x) < g(x_i) - f(x) < g(x_i) - y_{i-1}. \tag{1}$$

Now we bound the terms on the left and right. Since

$$|g(x_{i-1}) - y_i| \le |g(x_{i-1}) - y_{i-1}| + |y_{i-1} - y_i| < \varepsilon \quad \text{and} \quad |g(x_i) - y_{i-1}| \le |g(x_i) - y_i| + |y_{i-1} - y_i| < \varepsilon,$$

then combining this with (1) shows that $|g(x) - f(x)| < \varepsilon$. Since this is true for any $x \in C$, then $\sup_{x \in C} |g(x) - f(x)| = \max_{x \in C} |g(x) - f(x)| < \varepsilon$, since C is compact. Since $g \in U$ was arbitrary, we have $U \subset B_C(f, \varepsilon)$. □