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Math 561
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Problem Set 7

Pr. 2.

Show that in the compact-open topology, $\mathcal{C}(X,Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular. [Hint: If $\overline{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$.]

Def : X, Y spaces. $C \subset X$ compact, $U \subset Y$ open.
Subbasis element $S(C,U) = \{f \in \mathcal{C}(X,Y) \text{ s.t. } f(C) \subset U\}$.

Proof :

(i) Suppose Y is Hausdorff.
Let $f, g \in \mathcal{C}(X,Y)$ such that $f \neq g$.
Then $\exists x \in X$ such that $f(x) \neq g(x)$.
Now, $f(x), g(x) \in Y$ and Y is Hausdorff,
Therefore, $\exists U, V \subset Y$ open-disjoint such that $f(x) \in U$ and $g(x) \in V$.
Then $f \in S(\{x\}, U)$ and $g \in S(\{x\}, V)$, where
$S(\{x\}, U)$ and $S(\{x\}, V)$ are disjoint-open sets in $\mathcal{C}(X,Y)$ with compact-open topology.
Therefore, $\mathcal{C}(X,Y)$ is Hausdorff.

(ii) Suppose Y is regular.
Let $f \in \mathcal{C}(X,Y)$ and $\mathcal{V} \subset \mathcal{C}(X,Y)$ be an open neighborhood of f (we may assume that $\mathcal{V}$ is a basic neighborhood).
$\mathcal{V} = S(C_1, V_1) \cap \cdots \cap S(C_n, V_n)$ where $C_i \subset X$ compact, $V_i \subset Y$ open, and $f(C_i) \subset V_i$.

Goal : to find a neighborhood $\mathcal{U} \subset \mathcal{C}(X,Y)$ of f such that $f \in \mathcal{U} \subset \overline{\mathcal{U}} \subset \mathcal{V}$.

Since Y is regular, for each $i = 1, \cdots, n$ we may find a neighborhood $U_i \subset Y$ so that $f(C_i) \subset U_i \subset \overline{U_i} \subset V_i$, because $f(C_i) \subset Y$ is compact by the continuity of f.

Now, Let $\mathcal{U} = S(C_1, U_1) \cap \cdots \cap S(C_n, U_n)$.
Then $\mathcal{U} \subset \mathcal{C}(X,Y)$ is an open neighborhood of f.

Let $\mathcal{F} = \{g \in \mathcal{C}(X,Y); g(C_1) \subset \overline{U}_1, \cdots, g(C_n) \subset \overline{U}_n\}$. Then clearly $\mathcal{U} \subset \mathcal{F}$.

Claim : (a) $\mathcal{F} \subset \mathcal{V}$, (b) $\mathcal{F}$ is closed in $\mathcal{C}(X,Y)$.
Then we will have $\overline{\mathcal{U}} \subset \mathcal{F} \subset \mathcal{V}$, and we will be done.

(a) Since $\overline{U}_i \subset V_i$ for each $i = 1, \cdots, n$, it is clear that $\mathcal{F} \subset \mathcal{V}$.

(b) We show that the complement of $\mathcal{F}$ is open.
Let $g \in \mathcal{C}(X,Y) \setminus \mathcal{F}$.
Then $g(C_{i_0}) \not\subset \overline{U}_{i_0}$ for some i_0.
So, $\exists x_0 \in C_{i_0}$ such that $g(x_0) \notin \overline{U}_{i_0}$.
Let $W \subset Y$ be a neighborhood of $g(x_0)$ disjoint from $\overline{U}_{i_0}$.
Then $S(x_0, W)$ is a neighborhood of g, and if $h \in S(x_0, W)$, then $h(C_{i_0}) \not\subset \overline{U}_{i_0}$.
Therefore, $h \notin \mathcal{F}$. Hence the complement of $\mathcal{F}$ is open.

Let C be a compact subset of a regular space X. Let V be an open cover of C. Then we can find an open cover U of C such that $C \subset U \subset \overline{U} \subset V$.

Proof: V^c is closed in X such that $x \notin V^c$ for every $x \in C$.
Then for each $x \in C$ we can find disjoint-open neighborhoods W_x and U_x s.t.
$V^c \subset W_x$ and $x \in U_x$.

Now, $C \subset \bigcup_{x \in C} U_x$.
So, using the compactness of C, we can find finitely many of such $x's$ such that
$C \subset U = \bigcup_{i=1}^{n} U_{x_i}$.
And also $V^c \subset W = \bigcap_{i=1}^{n} W_{x_i}$
So, $U \cap W = \emptyset$
Then $U \subset W^c$ and W^c is closed, so $\overline{U} \subset W^c \subset V$.