Topology Problem 7.6

Sam Wilson

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Problem: The Trigonometric Polynomials

$$p(x) = \sum_{k=0}^{N} (a_k \cos kx + b_k \sin kx)$$

is dense in the space of 2π periodic functions from $\mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence. Since the functions are periodic with $T = 2\pi$, it suffices to show the conditions on the interval $[0, 2\pi)$.

Proof. Let $\mathcal{A}$ be the function algebra generated by $1, \cos(x), \sin(x)$. We will show that this algebra satisfies the conditions of the Stone-Weierstrass theorem.

Separates Points: Let $x_1, x_2 \in [0, 2\pi)$ such that $x_1 \neq x_2$. Thus, $\sin(x)$ is a function is $\mathcal{A}$ such that $\sin(x_1) \neq \sin(x_2)$. Therefore $\mathcal{A}$ separates points.

Nowhere Vanishing: Let $a \in [0, 2\pi)$. If $\sin(a) = 0$, then $\cos(a) \neq 0$. Otherwise, $\sin(a) \neq$ Thus, for every point $a \in \mathbb{R}$, there exists a function $f \in \mathcal{A}$ such that $f(a) \neq 0$

Therefore, by the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in the space of 2π periodic functions from $\mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence. □