Topology Problem 7.6

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Problem: The set of trigonometric polynomials

$$p(x) = \sum_{k=0}^{N} (a_k \cos kx + b_k \sin kx)$$

is dense in the space of 2π periodic functions from $\mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence. Since the functions are periodic with $T = 2\pi$, it suffices to show the conditions on the interval $[0, 2\pi)$. Thus, we will work in the space $X = [0, 2\pi)$ with the topology of uniform convergence of continuous real-valued functions.

Proof. Let $\mathcal{A}$ be the function algebra generated by $\cos(kx), \sin(kx), \forall k \in \mathbb{Z}$ [a short argument that this vector space is an algebra below]. We will show that this algebra satisfies the conditions of the Stone-Weierstrass theorem.

Separates Points: Let $x_1, x_2 \in [0, 2\pi)$ such that $x_1 \neq x_2$. Note that if $\sin(x_1) = \sin(x_2)$, then certainly $\cos(x_1) \neq \cos(x_2)$. Thus, for every pair of points, there exists a function in our algebra such that $f(x_1) \neq f(x_2)$

Nowhere Vanishing: Let $a \in [0, 2\pi)$. If $\sin(a) = 0$, then $\cos(a) \neq 0$. Otherwise, $\sin(a) \neq 0$ Thus, for every point $a \in \mathbb{R}$, there exists a function $f \in \mathcal{A}$ such that $f(a) \neq 0$

Therefore, by the Stone-Weierstrass Theorem, $\mathcal{A}$ is dense in the space of 2π periodic functions from $\mathbb{R} \to \mathbb{R}$, with the topology of uniform convergence. □

Lemma. *The vector space spanned by* $\cos(kx), \sin(kx), \forall k \in \mathbb{R}$ *is a function algebra*

Proof. We need to show that products of functions in our space is also a function in our space. To do so, note the following trigonometric identities for all $n \in \mathbb{N}$:

$$(1) \quad \sin^{2n}(x) = \frac{1}{2^{2n}}\binom{2n}{2} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos(2(n-k)x)$$

$$(2) \quad \sin^{2n+1}(x) = \frac{(-1)^4}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin((2k+1-2k)x)$$

$$(3) \quad \cos^{2n}(x) = \frac{1}{2^{2n}}\binom{2n}{2} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos(2(n-k)x)$$

$$(4) \quad \cos^{2n+1}(x) = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{k}\cos((2k+1-2k)x)$$

Using these identities (along with $\sin(2x) = \sin(x)\cos(x)$), we can turn any product of functions of the form $\cos(kx), \sin(kx), \forall k \in \mathbb{R}$ into that of a similar form solely made from linear combinations of $\cos(kx), \sin(kx)$. Thus, it follows that products functions in the space are also functions in the space, meaning that the vector space must be an algebra. □