MATH 561 - PROBLEM SET 7

PATRICK GILLESPIE

Problem 8. The space $\mathcal{F}_p(\mathbb{R}, \mathbb{R})$ of all functions (with the topology of pointwise convergence) is not compactly generated.

Solution. Let $T \subseteq \mathcal{F}_p(\mathbb{R}, \mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = n$ for all but at most n values of x, and $f(x) = 0$ otherwise, where $n \in \mathbb{N}$. We show that T is not closed in $\mathcal{F}_p(\mathbb{R}, \mathbb{R})$, yet the intersection, $T \cap K$, of T with any compact set K is closed in K.

T is not a closed subset of $\mathcal{F}_p(\mathbb{R}, \mathbb{R})$. Consider the zero function $g = 0$, i.e. $g(x) = 0$ for all $x \in \mathbb{R}$. A basic open neighborhood of g is of the form $U = \{f : \mathbb{R} \to \mathbb{R} \mid f(x_i) \in U_i, i \in \{1, \ldots, m\}\}$ for some choice of $x_i \in \mathbb{R}$ and neighborhoods U_i of $0 \in \mathbb{R}$ for all $i \in \{1, \ldots, m\}$. However, for any such neighborhood U of g, take the element $f' \in T$ defined by $f'(x_i) = 0$ for all $i \in \{1, \ldots, m\}$ and $f'(x) = m$ otherwise, where $x_1, \ldots, x_m$ are the elements used to define U. Then $f' \in U$. Hence g is a limit point of T, but not an element of T. Thus T is not closed.

However, T_N is closed, where we define T_N to be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = n$ for all but at most n values of x, and $f(x) = 0$ otherwise, where $n \leq N$. This can be seen by the fact that if $f \notin T_N$, one of the following conditions hold:

1. there exists $x \in \mathbb{R}$ such that $f(x) \notin \{0, 1, 2, \ldots, N\}$
2. there exist $x, y \in \mathbb{R}$ such that $f(x), f(y) \in \{1, 2, \ldots, N\}$ and $f(x) \neq f(y)$
3. $f(x) = n \in \{1, 2, \ldots, N\}$ for some $x \in \mathbb{R}$ and $f(x) = 0$ for more than n values of x
4. $f(x) = 0$ for all $x \in \mathbb{R}$.

In either case, we can easily find a neighborhood of f disjoint from T_N.

To show that $T \cap K$ is closed in K whenever K is compact, we first show that for any compact set $K \subset \mathcal{F}_p(\mathbb{R}, \mathbb{R})$, we have that $\{f(x) \mid f \in K\}$ is bounded for any fixed $x \in \mathbb{R}$. So suppose K is compact and fix $x_0 \in \mathbb{R}$. Consider the open cover $\mathscr{U}$ of K whose elements are the open sets of the form $K \cap \{f : \mathbb{R} \to \mathbb{R} \mid f(x_0) \in (n, n+2)\}$ where $n \in \mathbb{N}$. Since K is compact, $\mathscr{U}$ admits a finite subcover. But then this clearly implies that $\{f(x_0) \mid f \in K\}$ is bounded.

Now if K is compact, by the previous paragraph, we see that $T \cap K \subset T_N \cap K$ for some $N \in \mathbb{N}$, and hence $T \cap K = T_N \cap K$. Since $\mathcal{F}_p(\mathbb{R}, \mathbb{R})$ is Hausdorff, K is closed. Then since T_N is closed, $T_N \cap K = T \cap K$ is closed. In particular, $T \cap K$ is closed in K. Therefore T is not closed, yet $T \cap K$ is closed in K for all compact K. Hence $\mathcal{F}_p(\mathbb{R}, \mathbb{R})$ is not compactly generated.

□

Date: October 30, 2020.