Topology Homework 7

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9 Let X be locally compact Hausdorff. If X is σ-compact with compact exhaustion $(K_n)_{n\geq 1}$, define a metric on $C(X)$ by

$$\rho(f,g) = \sum_{n=1}^{\infty} \rho_n(f,g)$$
$$\rho_n(f,g) = \min\{\frac{1}{2^n}, \sup_{x\in K_n} |f(x) - g(x)|\}$$

Show that ρ metrizes the u.o.c. topology on $C(X)$.

Proof. We need to show that a ball in ρ contains a ϵ-ball in the u.o.c. topology and vice-versa. In other words, we need to show that for $R > 0$, $U(f,R) = \{g \mid \rho(f,g) < R\}$ contains some $B_C(f,\epsilon) = \{g \mid d(f(x), g(x)) < \epsilon, \forall x \in C\}$ for some compact $C \subset X$ and $\epsilon > 0$.
Since X is exhausted by $(K_n)_{n\geq 1}$, we know that the compact set $C \subset X$ will be contained in K_m for some m. If we choose $\epsilon = \rho_m(f,g)$, we'll have the ball $B_C(f, \rho_m(f,g)) = \{g \mid d(f(x), g(x)) < \rho_m(f,g)\}$, which will be contained in the ball $U(f,R)$.

To see that any ϵ-ball contains a ball in ρ, let $\epsilon > 0$. Since $B_C(f,\epsilon)$ is defined $\forall x \in C$, we know C is contained in some K_m. Then we can choose $R = \rho_m(f,g)$. Then the ball $U(f,R)\{g \mid \rho(f,g) < \rho_m(f,g)\}$ will be contained in $B_C(f,\epsilon)$.

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