8 Connectedness and Continuous maps

11. If $f : X \to Y$ is a continuous map, and X is connected, then the graph $\Gamma_f \subset X \times Y$ is connected. Is the converse true?
Recall: X is connected, $f : X \to Y$(continuous and surjective) $\Rightarrow$ Y connected

Proof. Consider the map $F : X \to X \times Y$ given by $F(x) = (x, f(x))$. Since $f(x)$ is continuous, $F(x)$ is continuous, and its image equal to the graph of f. The graph of f is connected because the continuous image of a connected space is connected, .
The converse is not true in general.

Consider the topologist's sine curve : $\Gamma_f = 0 \cup S$ where $S = (x, \sin \frac{1}{x} : x \in (0, 1])$. S is the image of a path connected set $(0, 1]$ under a continuous map; hence it is connected and its closure $(0 \times [-1, 1]) \cup S$ is connected. Therefore , Γ_f is connected. However, the function:

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

is not continuous at $x = 0$. To see this, consider the sequence $x_n = \frac{2}{\pi(2n-1)}$. $x_n \to 0$ but $f(x_n) \to 1 \neq 0 = f(0)$ □