Topology HW 10.13

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13 A connected metric space with more than one point is uncountable.

Proof. Let (X, d) be a connected metric space, and assume we have $x_0, x_1 \in X$ with $x_0 \neq x_1$. We can define a function $f : X \to [0, 1]$ in terms of the metric:

$$f(x) = \frac{d(x, x_1)}{d(x, x_0) + d(x, x_1)}$$

The continuity of f follows from the continuity of d. We can see also that $f(x_0) = 0$ and $f(x_1) = 1$. Thus, we have a continuous map of a connected space into the unit interval $[0, 1]$. By the Intermediate Value Theorem, we must have that for every $c \in (0, 1)$, there exists some $x \in X$ such that $f(x) = c$. Hence, X is uncountable.

$\square$