Topology Problem Set 8

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14. Show that the group $G(n)$ of $n \times n$ real matrices with nonzero determinant is not connected. Specifically, if $D : G(n) \to \mathbb{R} \setminus \{0\}$ is the determinant function, then the connected components of $G(n)$ are $G_+ = D^{-1}((0, \infty))$ and $G_- = D^{-1}((-\infty, 0))$.

Proof. First, note that the larger set $\mathcal{M}(n)$ of real $n \times n$ matrices which contain $G(n)$ is a n^2 dimensional real vector space with a natural topology inherited by considering it as a subset of $\mathbb{R}^{n^2}$ with the product topology. The determinant function, being a specific composition of the multiplication and addition operations on permutations of the coordinates, is clearly continuous on $\mathcal{M}(n)$. Thus, the sets G_+ and G_- certainly form a separation of $G(n)$. Thus it remains to show that each of these open sets are connected. We will do this by first defining $B_+ = \text{diag}(1, 1, \cdots, 1)$ and $B_- = \text{diag}(-1, 1, 1, \cdots, 1)$ and show that for any $M \in G_+$ and $N \in G_-$ there exist continuous functions $f : [0, 1] \to G_+$ and $g : [0, 1] \to G_-$ so $f(0) = M$, $g(0) = N$, $f(1) = B_+$, and $g(1) = B_-$. Note that, since the determinant of triangular matrices is just the product of the elements in its diagonal, we have $D(B_+) = 1$ and $D(B_-) = -1$.

Denote by, for $k, l \leq n$ with $k \neq l$ and some $\alpha \in \mathbb{R}$, R^{kl}_α the operation on $\mathcal{M}(n)$ which adds the multiple $\alpha[a_{l1}, a_{l2}, \cdots, a_{ln}]$ of row l of a matrix $(a_{ij}) \in \mathcal{M}(n)$ to row k. That is, if $(a_{ij}) \in \mathcal{M}(n)$ then $R^{kl}_\alpha((a_{ij})) = (b_{ij})$, where $b_{ij} = a_{ij}$ if $i \neq k$ and $b_{kj} = a_{kj} + \alpha a_{lj}$ otherwise. It is a basic fact of linear algebra that $D(R^{kl}_\alpha(A)) = D(A)$ for any matrix $A \in \mathcal{M}(n)$. Thus, the function $[0, 1] \to \mathcal{M}(n)$ given by $t \mapsto R^{kl}_{\alpha t}(A)$ is a continuous path in $\mathcal{M}$ whose in $\mathcal{M}(n)$ is mapped to a single point in $\mathbb{R}$ by D. This gives a convenient way to construct paths $[0, 1] \to G_+$ or $[0, 1] \to G_-$; just row-reduce to an appropriate diagonal matrix and then continuously dilate each entry on the diagonal to 1 or -1 to complete the path to B_+ or B_-.

I will attempt to describe part of process of "continuous row reduction" to B_- on a general $M = (a_{ij}) \in G_-$. The process is exactly analogous in G_+. First, we start with $i = 1$. First, we may continuously combine rows to make the first column

$$\begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix};$$

this process is fairly easy to conceptualize; I refer you to any introductory linear algebra book to the description of Gaussian reduction. If we were in G_+, this first element could simply be chosen to be 1. Again, these transformations do not change the determinant of the matrix at any point along the corresponding path in G_-. We continue on, making column i of M to

$$\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

with the 1 in the i^{th} row. What remains in the final column can be further reduced to

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ K \end{bmatrix},$$

where K is a positive number, in this case equal to $-D(M)$. If we were in G_+, then K would be equal to this determinant. To see this, note the form of the matrix is triangular, and so the determinant is the product of the elements along the diagonal. Finally, we form the final leg of the path by

$$t \longmapsto \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & K \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{t}{K} + (1-t) \end{bmatrix},$$

which is a scaling term which finally completes our journey to B_-. Thus, each space G_- and G_+ is in fact *path connected*. □