MATH 561 - PROBLEM SET 8

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Problem 16. (i) What are the components and path components of $\mathbb{R}^{\mathbb{N}}$, in the product topology?

(ii) Give $X = \mathbb{R}^{\mathbb{N}}$ the uniform topology. Show that $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ lie in the same component of X if and only if the sequence $\{x_i - y_i\}$ is bounded.

Solution. (i) We show that $\mathbb{R}^{\mathbb{N}}$ is path connected, hence $\mathbb{R}^{\mathbb{N}}$ is the single component and path component. Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ be elements of $\mathbb{R}^{\mathbb{N}}$. Since $\mathbb{R}$ is path connected, there exist paths $\alpha_i : [0,1] \to \mathbb{R}$ with $\alpha_i(0) = x_i$ and $\alpha_i(1) = y_i$ for all $i \in \mathbb{N}$. We now define the path $\alpha : [0,1] \to \mathbb{R}^{\mathbb{N}}$ by $\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots)$. Clearly, $\alpha(0) = \mathbf{x}$ and $\alpha(1) = \mathbf{y}$. Moreover, by the universal property of the product topology, since each α_i is continuous, α is continuous as well. Thus there exists a path from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbb{R}^{\mathbb{N}}$. Hence $\mathbb{R}^{\mathbb{N}}$ is path connected.

(ii) The self-map $X \to X$ defined by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$ for any fixed $\mathbf{a} \in X$ is a homeomorphism. Hence it suffices to show that $\mathbf{x} = (x_1, x_2, \dots) \in X$ and $\mathbf{0} = (0, 0, \dots) \in X$ lie in the same component of X if and only if the sequence $\{x_i\}$ is bounded. To prove the one direction, suppose that $\{x_i\}$ is bounded. We show that $\mathbf{x}$ lies in the same component as $\mathbf{0}$ by showing that the path $\alpha : [0,1] \to X$ from $\mathbf{0}$ to $\mathbf{x}$ defined by $\alpha(t) = t\mathbf{x}$ is continuous. Let $N \in \mathbb{N}$ be a bound for $\{x_i\}$. Fix $t \in [0,1]$ and let $\epsilon > 0$. Set $\delta = \epsilon/N$. Then for any $s \in [0,1]$ such that $|t - s| < \delta$, we have that

$$d(\alpha(t), \alpha(s)) = \sup_{i \in \mathbb{N}} |tx_i - sx_i| = \sup_{i \in \mathbb{N}} |(t-s)x_i| \le \sup_{i \in \mathbb{N}} |\delta x_i| = \sup_{i \in \mathbb{N}} |\frac{\epsilon x_i}{N}| \le \epsilon.$$

Thus $\mathbf{x}$ and $\mathbf{0}$ lie in the same path component, and hence component, of X.

For the other direction, suppose $\{x_i\}$ is not bounded. We show that the set $A = \{\mathbf{z} = (z_1, z_2, \dots) \in X \mid \{z_i\} \text{ is bounded}\}$ which contains $\mathbf{0}$ but not $\mathbf{x}$ is both closed and open. Let $\mathbf{z} \in A$ and consider the open ball $B(\mathbf{z}, 1) = \{\mathbf{z}' \in X \mid \sup_{i \in \mathbb{N}} |z_i - z_i'| < 1\}$. Clearly, if N is a bound for $\{z_i\}$, then $N + 1$ is a bound for $\{z_i'\}$, hence $\mathbf{z}' \in A$ for any $\mathbf{z}' \in B(\mathbf{z}, 1)$. Thus A is open. Similarly if $\mathbf{z} \notin A$, then $B(\mathbf{z}, 1) \subset A^c$, hence A is closed. Thus $\mathbf{x}$ and $\mathbf{0}$ do not lie in the same component of X. □

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