Problem Set 8

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November 17, 2020

Problem 17.
If Y is Hausdorff and connected (respectively, path-connected) and X is Hausdorff, is $\mathcal{F}(X,Y)$ with the pointwise topology connected (respectively, path-connected)? What about in the compact-open topology? (Consider the case $X = S^1$, the unit circle, and $Y = \mathbb{R}^2 \setminus \{0\}$.)

Proof.

- We show that $\mathcal{F}(X,Y)$ with the pointwise topology is path-connected if Y is path-connected. Let $f, g \in \mathcal{F}(X,Y)$. Since Y is path-connected, for every $x \in X$, there exists a continuous map $\gamma_x : [0,1] \to Y$ such that $\gamma_x(0) = f(x)$ and $\gamma_x(1) = g(x)$.

  Define $\gamma : [0,1] \to \mathcal{F}(X,Y)$ by $\gamma(t) = \gamma_x(t)$, where the result is a function of x, for each t. It is easy to see that $\gamma(0) = f$ and $\gamma(1) = g$.

  We show that γ is continuous. If U is a basic open set in $\mathcal{F}(X,Y)$, then $U = (U_x)$ where $U_x \neq Y$ for only finitely many $x \in X$. Denote this subset of finitely many elements of X by F. Then

  $$\gamma^{-1}(U) = \gamma^{-1}\left(\prod_{x\in X} U_x\right) = \bigcap_{x\in X} \gamma_x^{-1}(U_x) = \bigcap_{x\in F} \gamma_x^{-1}(U_x)$$

  which is a finite intersection of open sets, as each $\gamma_x^{-1}(U_x)$ is open by continuity of γ_x.

  Therefore $\mathcal{F}(X,Y)$ is path-connected.

- Next we show that $\mathcal{F}(X,Y)$ with the pointwise topology is connected if Y is connected. Assume that $\mathcal{F}(X,Y)$ is not connected. Then we can write $\mathcal{F}(X,Y) = A \sqcup B$, where A, B are both nonempty, open subsets of $\mathcal{F}(X,Y)$. Since $\mathcal{F}(X,Y)$ is endowed with the product topology, projection maps are open, so $\pi_x(A)$ and $\pi_x(B)$ are open in Y for every $x \in X$.

  We show that there is an $x \in X$ such that $\pi_x(A) \sqcup \pi_x(B) = Y$.

  First, note that there exists an $x \in X$ such that $\pi_x(A) \cap \pi_x(B) = \emptyset$, as otherwise, we could find $y_x \in \pi_x(A) \cap \pi_x(B)$ for each $x \in X$, so there is a function $f \in A$ such that $f(x) = y_x$ for each x and there is a function $g \in B$ such that $g(x) = y_x$ for each x. But then $f = g$, so we have that $A \cap B \neq \emptyset$, a contradiction.

  Next, for this x, we have that $\pi_x(A) \cup \pi_x(B) = Y$. Certainly the forward containment holds. If $y \in Y$, then $\exists f \in \mathcal{F}(X,Y)$ such that $f(x) = y$. Since $\mathcal{F}(X,Y) = A \sqcup B$, $f \in A$ or $f \in B$. Thus $y \in \pi_x(A)$ or $y \in \pi_x(B)$, so $y \in \pi_x(A) \cup \pi_x(B)$.

  Lastly, we note that since A and B are nonempty, their projections $\pi_x(A)$ and $\pi_x(B)$ are also nonempty. Therefore we have written $Y = \pi_x(A) \sqcup \pi_x(B)$, both open, disconnecting Y, a contradiction.

- Through correspondence with Dr. Freire, I have realized the compact-open topology questions deal with homotopies, which we have not covered yet.

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