Math 561, Topology

Set 8

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Problem 2 (i): If each X_α is connected and $\bigcap_\alpha X_\alpha \neq \emptyset$, then $X = \bigcup_\alpha X_\alpha$ is connected.

Solution:
Let U and V open and disjoint such that $X = U \cup V$. Then

$$\bigcap_\alpha X_\alpha \subset U \cup V.$$

Take x in the intersection and assume that $x \in U$, then for every X_α we have that $X_\alpha \subset U \cup V$ with U and V open and disjoint and $X_\alpha \cap U \neq \emptyset$. But each X_α is connected, hence $V \cap X_\alpha$ must be empty for every α, which implies that V itself is empty. Since these open set were arbitrary we can conclude that X has no separations and thus is connected. ■

Problem 2 (ii): If each pair of points $x, y \in X$ lie in some connected subset $E_{xy} \subset X$, then X is connected.

Solution:
Fix $x \in X$ and for $y \in X$ let $X_y = E_{xy}$ some connected subset of X containing both points. Then

$$x \in \bigcap_{y \in X} X_y \qquad \text{and} \qquad \bigcup_{y \in X} X_y = X.$$

Hence by the previous part, X is connected. ■

Problem 2 (iii): If $X = \bigcup_{n \geq 1} X_n$, each X_n is connected and $X_n \cap X_{n+1} \neq \emptyset$ for all n, then X is connected.

Solution:
Consider the sets $Y_n = \bigcup_{k=1}^n X_k = Y_{n-1} \cup X_n$. Notice first that $Y_n \nearrow X$, hence for any pair of points x and y in X there is some $N \in \mathbb{N}$ such that $x, y \in Y_N$. Moreover, $Y_1 = X_1$ is connected and for $n > 1$ we have

$$\emptyset \neq X_{n-1} \cap X_n \subseteq Y_{n-1} \cap X_n.$$

So using induction and problem (i) we can conclude that each Y_n is connected. But then by problem (ii) we have that X is also connected. ■