Topology Problem 8.6

Sam Wilson

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Problem: Let X be a connected space and let $\mathcal{U}$ be an open cover for X. Any two points $a, b \in X$ can be connected with a chain of open sets belonging to $\mathcal{U}$.

Proof. Let R be the set of points, $r \in X$ such that r can be connected to b by a chain of open sets belonging to $\mathcal{U}$.

Open and Non-empty: Note that since $b \in R$, R is nonempty. Also, note that each point in an open chain $G_1, G_2, \ldots, G_n$ is also in R. Thus, R is a neighborhood of each of its points so R is open.

Closed: Let $p \in X \setminus R$, and let $U_p \in \mathcal{U}$ be an open set containing p. Now, if $U_p \cap R \neq \varnothing$, then there exists a point $q \in U_p$ and $q \in R$. Thus, there exists a open chain such that p connects to b via q. This contradicts the fact that $p \in X \setminus R$. Thus, $U_p \cap R = \varnothing$. This means $X \setminus R$ is a neighborhood of each of its points, and thus is open, which means $X \setminus (x \setminus R)$ is closed, but $X \setminus (X \setminus R) = R$. Thus, R is a clopen set that is nonempty, and since X is connected, it follows that $R = X$. □