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Math 561
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Problem Set 8

Pr. 9.

The unit sphere in a normed vector space (of dimension at least 2) is path connected.

Proof :

Let X be a normed vector space with $\dim(X) \geq 2$.
Let $S = \{x \in X; \|x\| = 1\}$.

Let $x, y \in X$. Define $f : [0,1] \to S$ by,

$$f(t) := \frac{(1-t)x + ty}{\|(1-t)x + ty\|}, \qquad t \in [0,1]$$

Notice that f is continuous for $(1-t)x + ty \neq 0$.

$$\text{So, if } (1-t)x + ty \neq 0, \text{ then } f(0) = \frac{x}{\|x\|} = x, \qquad \|x\| = 1$$

$$f(1) = \frac{y}{\|y\|} = y, \qquad \|y\| = 1$$

Then f is a path in S joining x and y $\qquad (\star)$

Otherwise, if $(1-t)x + ty = 0$, then

$$\begin{aligned} (1-t)x &= ty \\ \|(1-t)x\| &= \|ty\| \\ 1-t &= t, \qquad \|x\|, \|y\| = 1 \\ t &= \frac{1}{2} \end{aligned}$$

Then, $(1-t)x + ty = 0$, gives us $\frac{1}{2}x + \frac{1}{2}y = 0$, or $x = -y$

Which means, x and y are antipodal points.

Now, since $\dim(X) \geq 2$, we can find $z \in S$ different from x and y. Hence, z is not an antipodal point of x or y.

Then by $(\star)$, there exist paths in S joining x and z, and z and y.
Let $f : [0,1] \to S$ be a path joining x and z, and $g : [1,2] \to S$ be a path joining z and y.
Then, $h : [0,2] \to S$ defined by,

$$h(t) = \begin{cases} f(t) & 0 \leq t \leq 1 \\ g(t) & 1 \leq t \leq 2 \end{cases}$$

is continuous by the pasting lemma, and is a path in S joining x and y.