Problems 3.5-7

Jacob Honeycutt

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5. (i) X Hausdorff, $K \subseteq X$ is compact $\implies$ K is closed in X.
   (ii) X compact Hausdorff, $C \subseteq X$ closed $\implies$ C is compact.
   (iii) X, Y Hausdorff, X compact, $f : X \to Y$ continuous, injective $\implies$ f is an embedding.

Proof. (i) Let X be Hausdorff and $K \subseteq X$ be compact. We show that $X \setminus K$ is open. To do so, let $x \in X \setminus K$. Since X is Hausdorff, for any $y \in K$, $\exists U_y, V_y$ open in X with $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. As y ranges over all of K, $\{V_y\}$ is an open cover of K.

Since K is compact, there is a finite subcover, say $\{V_{y_1}, \ldots, V_{y_n}\}$. Take

$$U = \bigcap_{i=1}^{n} U_{y_i}.$$

This is open, as it is a finite intersection of open sets. Further, since $x \in U_{y_i}$ for all i, $x \in U$. We show that U is disjoint from K. Since $\{V_{y_1}, \ldots, V_{y_n}\}$ is an open cover of K, we have

$$K \subseteq \bigcup_{i=1}^{n} V_{y_i}.$$

Hence if $z \in U$, then $z \in U_{y_i}$ for all i, so $z \notin V_{y_i}$ for any i. Thus

$$z \notin \bigcup_{i=1}^{n} V_{y_i},$$

giving $z \notin K$. From this, $x \in U \subseteq X \setminus K$, so $X \setminus K$ is open. Therefore K is closed, as desired.

(ii) Let X be compact Hausdorff and $C \subseteq X$ be closed. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of C.
Then $\{U_\alpha\}_{\alpha \in \Lambda} \cup \{X \setminus C\}$ is an open cover of X, and since X is compact, there is a finite subcover $\{U_1, \ldots, U_n\} \cup \{X \setminus C\}$, where $X \setminus C$ may or may not need to be in the subcover.
In any case, $\{U_1, \ldots, U_n\}$ is a finite subcover of $\{U_\alpha\}_{\alpha \in \Lambda}$, showing that C is compact.

(iii) Let X, Y be Hausdorff, X be compact, $f : X \to Y$ be continuous and injective. All that we need to do is show that f is open or closed, then we get that f is an embedding. Let $F \subseteq X$ be closed. By (ii) above, F is compact. Then by 5.5 below, $f(F)$ is compact in Y. Since $f(F)$ is compact, by (i) above, it is closed in Y. Therefore f is a closed map, and hence an embedding into Y.

□

5.5. $C \subseteq X$ a compact subset, $f : X \to Y$ continuous $\implies$ $f(C)$ is a compact subset of Y.

Proof. Let $C \subseteq X$ be a compact subset and $f : X \to Y$ be continuous. Let $\{U_i\}_{i \in I}$ be an open cover of $f(C)$. Then since f is continuous, $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of C.

Since C is compact, $\exists f^{-1}(U_1), \ldots, f^{-1}(U_n)$ such that $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ is a finite subcover. Then $\{U_1, \ldots, U_n\}$ is a finite subcover of $\{U_i\}_{i \in I}$. Thus $f(C)$ is compact in Y. □

6. (i) X is compact iff any family $\{C_\lambda\}_{\lambda \in A}$ of closed subsets of X with the *finite intersection property* has nonempty intersection:

$$\left(\bigcap_{\lambda \in F} C_\lambda \neq \emptyset, \quad \forall F \subseteq A \text{ finite} \right) \implies \bigcap_{\lambda \in A} C_\lambda \neq \emptyset.$$

(ii) $K_1 \supseteq K_2 \supseteq \cdots$ compact and nonempty $\implies$

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Proof. (i) ($\implies$) Assume X is compact. Let $\{C_\lambda\}_{\lambda \in A}$ be a collection of closed sets with the finite intersection property. Assume to the contrary that

$$\bigcap_{\lambda \in A} C_\lambda = \emptyset.$$

Taking complements, we have that

$$\bigcup_{\lambda \in A} (X \setminus C_\lambda) = X.$$

But since $X \setminus C_\lambda$ are open, this gives an open cover of X, which has a finite subcover, say $\{X \setminus C_\lambda\}_{\lambda \in F}$ for $F \subseteq A$ finite such that

$$\bigcup_{\lambda \in F} (X \setminus C_\lambda) = X.$$

But taking complements again gives

$$\bigcap_{\lambda \in F} C_\lambda = \emptyset,$$

violating the finite intersection property.

($\impliedby$) Assume that for any collection of closed sets with the finite intersection property, the intersection of the collection is nonempty. Assume to the contrary that X is not compact. Then $\exists \{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of X such that $\forall F \subseteq \Lambda$ finite subsets, $\{U_\alpha\}_{\alpha \in F}$ is not a cover of X. But this means that

$$\bigcup_{\alpha \in \Lambda} U_\alpha = X$$

and

$$\bigcup_{\alpha \in F} U_\alpha \neq X \quad \forall F \subseteq \Lambda \text{ finite}.$$

Taking complements, we get

$$\bigcap_{\alpha \in \Lambda} (X \setminus U_\alpha) = \emptyset$$

and

$$\bigcap_{\alpha \in F} (X \setminus U_\alpha) \neq \emptyset \quad \forall F \subseteq \Lambda \text{ finite},$$

but this is a contradiction to our original assumption.

(ii) Let $K_1 \supseteq K_2 \supseteq \cdots$ be nonempty compact sets. Assume to the contrary that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

Then taking complements (in K_1) gives

$$\bigcup_{n=1}^{\infty}(K_1 \setminus K_n) = K_1,$$

which yields an open cover of K_1. As K_1 is compact, there is a finite subcover, say $\{K_1 \setminus K_1, \ldots, K_1 \setminus K_n\}$. Hence

$$\bigcup_{i=1}^{n}(K_1 \setminus K_n) = K_1.$$

But taking complements again gives

$$\bigcap_{i=1}^{n} K_n = \emptyset,$$

a contradiction since these are nonempty nested sets.

□

7. Compact Hausdorff spaces are normal.

Proof. Let X be compact Hausdorff. First we show that X is regular. Let $A \subseteq X$ be closed and let $b \in X \setminus A$. By Hausdorff, for each $a \in A$, $\exists U_a, V_a$ open in X such that $a \in U_a$, $b \in V_a$, and $U_a \cap V_a = \emptyset$.

Then $\{U_a\}_{a \in A}$ is an open cover of A, and since A is a closed subset of a compact set X, it is too compact. Because of this, here is a finite subcover of $\{U_a\}_{a \in A}$, say $\{U_1, \ldots, U_n\}$. Defining

$$U = \bigcup_{i=1}^{n} U_i$$

and

$$V = \bigcap_{i=1}^{n} V_i,$$

where $\{V_1, \ldots, V_n\}$ are the V_a corresponding to the U_a in $\{U_1, \ldots, U_n\}$. We have that $A \subseteq U$ and $b \in V$, so we must show that $U \cap V = \emptyset$.

Let $x \in U$. Then $x \in U_i$ for some i, so by construction, $x \notin V_i$, since U_i and V_i are disjoint. This shows that $x \notin U$, and that $U \cap V = \emptyset$.

Now we show that X is normal. Let A, B be closed sets in X such that $A \cap B = \emptyset$. For each $b \in B$, apply the regularity of X to get U_b, V_b open with $A \subseteq U_b$, $b \in V_b$, and $U_b \cap V_b = \emptyset$. $\{V_b\}_{b \in B}$ is an open cover of B, which is compact, so there is a finite subcover $\{V_1, \ldots, V_n\}$. Defining

$$V = \bigcup_{i=1}^{n} V_i$$

and

$$U = \bigcap_{i=1}^{n} U_i,$$

where again the U_i are the U_b corresponding to the V_b in $\{V_1, \ldots, V_n\}$. Then $A \subseteq U$, $B \subseteq V$, and $A \cap B = \emptyset$ because if $x \in U$, then $x \in U$ for all i, giving $x \in V_i$ for no i, hence $x \notin V$. Therefore $U \cap V = \emptyset$ and X is normal. □