Problem Set 7

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12. (a) Exhibit a countable dense subset of $\mathcal{F}(I,I)$ with the pointwise topology. ($I = [0,1]$.)
    (b) Is $\mathcal{F}(I,I)$ separable with the topology of uniform convergence?

Proof. (a) Recall that $\mathcal{F}(I,I)$ is the set I^I endowed with the product topology. Consider the set of polynomials with rational coefficients on $[0,1]$. This is certainly a countable set.
Given a function $f \in \mathcal{F}(I,I)$, we take an open neighborhood U of f. As this is the product topology on I^I, we can find a basic set B such that $f \in B \subseteq U$. In particular, we can find $\varepsilon > 0$ small enough such that $f \in (B_x)_{x\in I} \subseteq B$ where $B_x = (f(x)-\varepsilon, f(x)+\varepsilon)$ for finitely many x, and $B_x = I$ otherwise. We can then choose a polynomial which interpolates points in each of these finiely many intervals (i.e. $p(x) \in (f(x)-\varepsilon, f(x)+\varepsilon)$), so the set of polynonmials with rational coefficients is also dense in $\mathcal{F}(I,I)$.

(b) No. Recall that any separable space cannot have an uncountable discrete subset. Note that if we take the collection of characteristic functions $\{\chi_A : A \subseteq [0,1]\}$. Note that for $A \neq B$,

$$\rho(\chi_A, \chi_B) = \sup_{x\in[0,1]} |\chi_A(x) - \chi_B(x)| = 1,$$

so $\{\chi_A : A \subseteq [0,1]\}$ is an uncountable discrete set.

□

14. Prove that the subspace $\mathbb{P} \subseteq \mathbb{R}$ of irrational numbers is homeomorphic to $\mathcal{F}_p(\mathbb{N},\mathbb{N})$ (topology of pointwise convergence). *Hint*: continued fraction expansions.

Proof. Recall that every irrational number has a unique continued fraction expansion, represented by $[a_0; a_1, a_2, \ldots]$ where $\{a_n\} \subseteq \mathbb{N}$. Using this, we create the function $f : \mathbb{P} \to \mathbb{N}^{\mathbb{N}}$ by

$$f(x) = (a_n)_{n\in\mathbb{N}}, \qquad x = [a_0; a_1, a_2, \ldots].$$

We show this is a homeomorphism.

Let $x \neq y \in \mathbb{P}$. As continued fraction expansions are unique, there is an $n \in \mathbb{N}$ such that $a_n \neq b_n$ where $x = [a_0; a_1, a_2, \ldots]$ and $y = [b_0; b_1, b_2, \ldots]$. Then $f(x) = (a_n) \neq (b_n) = f(y)$, so f is 1-1.

Let $(a_n) \in \mathbb{N}^{\mathbb{N}}$. Then there is an irrational x such that $x = [a_0; a_1, a_2, \ldots]$, so $f(x) = (a_n)$. Hence f is onto.

As both $\mathbb{P}$ and $\mathbb{N}^{\mathbb{N}}$ are metrizable, we show f preserves sequential convergence in both directions.

Suppose $x_m \to x \in \mathbb{P}$. For each $N \in \mathbb{N}$, we can find $M \in \mathbb{N}$ such that $\forall m \geq M$, $|x_m - x| < 10^{-N}$. Then x_m and x agree in the first $N-1$ decimal places, so the first $N-1$ digits of the continued fraction expansions agree as well. I.e. $\forall m \geq M$, $a_n^{(m)} = a_n$ $\forall n < N$. Taking $N \to \infty$, we see that $\left(a_n^{(m)}\right) \to (a_n)$.

Suppose now that $\left(a_n^{(m)}\right) \to (a_n)$ in $\mathbb{N}^{\mathbb{N}}$. Since $\left(a_n^{(m)}\right) \to (a_n)$, for any $N \in \mathbb{N}$, we can find $M \in \mathbb{N}$ such that $\forall m \geq M$, $a_n^{(m)} = a_n$ $\forall n < N$. Then the first $N-1$ digits of the continued fraction expansions agree, so the first $N-1$ decimal digits of x_m and x agree. Therefore $|x_m - x| < 10^{-N}$, so taking $N \to \infty$ gives that $x_m \to x$.

Thus f is a homeomorphism of $\mathbb{P}$ and $\mathbb{N}^{\mathbb{N}}$. □