Math 561, Topology

Set 7

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**Problem 19 (Already proven):** $(i)$ If $Y_1$, $Y_2$, ... are sequentially compact, then $Y = \prod Y_n$ is sequentially compact.

$(ii)$ If $N$ is countable and $Y$ is sequentially compact, $\mathcal{F}_p(N, Y)$ is sequentially compact.

**Problem 20:** Let $X \subseteq \mathbb{R}$ be arbitrary, $f_n : X \to [a, b]$ a sequence of monotone functions. Then $f_n$ has a convergent subsequence (pointwise in $X$).

**Solution:**

Since $\mathbb{R}$ is separable, so is $X$ and thus we can find a countable set $N$ that is dense in $X$. On the other hand, any closed and bounded interval in $\mathbb{R}$ is compact, and since $\mathbb{R}$ is a metric space it is also sequentially compact. Therefore $\mathcal{F}_p(N, [a, b])$ is sequentially compact by problem 19. So $f_n|_N$ has a convergent subsequence, we may assume without loss of generality that the sequence itself converges.

But notice that the limit, say $g$, is a function on $N$, hence we need to extend it to a function on $X$. First notice that by monotonicity of each $f_n$ we must have that $g$ is monotone as well (recall that limits preserve inequalities). Notice however that bounded monotone functions can only have countably many discontinuities, and they must all be jump discontinuities. Furthermore, $N$ is dense in $X$ and $X$ is a subset of $\mathbb{R}$. Thus we can extend $g$ by continuity approximating points in $X$ with points in $N$: for $x \in X$ with $x_n \in N$ such that $x_n \to x$ define $g(x) = \lim g(x_n)$. The limit is well defined except possibly at a countable set $M$ because $g$ is monotone.

For $x \in X \setminus M$ and any $u, v \in N$ satisfying $u \leq x \leq v$ we have that $f_n(u) \leq f_n(x) \leq f_n(v)$ (or $f_n(u) \geq f_n(x) \geq f_n(v)$) for every $n$. Hence,

$$g(u) \leq \liminf_{n\to\infty} f_n(x) \leq \limsup_{n\to\infty} f_n(x) \leq g(v).$$

Taking the limit as $u, v \to x$ gives then

$$g(x) \leq \liminf_{n\to\infty} f_n(x) \leq \limsup_{n\to\infty} f_n(x) \leq g(x).$$

So $f_n(x) \to g(x)$ for all $x \in X \setminus M$.

Finally, $M$ is countable and just like with $N$, problem 19 implies that $f_n|_M$ has a convergent subsequence $f_{n_k}|_M$ that converges to $g' : M \to [a, b]$. Define then $f : X \to [a, b]$ by

$$f = g\chi_{X\setminus M} + g'\chi_M.$$

From the above we can conclude that $f_{n_k}$ converges to $f$ pointwise in $X$. ■

**Problem 23:** If $M_i$ are compact metric spaces, then the product $\prod M_i$ is a compact metric space.

**Solution:**

Recall that sequential compactness is equivalent to compactness in metric spaces. Thus each $M_i$ is sequentially compact and by problem 19 $\prod M_i$ is sequentially compact.

On the other hand, each $M_i$ is a metric space with some distance $d_i$ which we may assume without loss of generality is bounded by 1 ($\min\{d_i, 1\}$ is also a metric and it is equivalent to $d_i$). We wish to prove that the distance function $d$ on $\prod M_i$ defined by

$$d(x,y) = \sup_{i \geq 1} \left\{ \frac{d_i(x_i, y_i)}{i} \right\}$$

is a metric that induces the product topology.

Indeed since each $d_i$ is a metric, the only not-so-trivial metric property of $d$ is the triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$. However, notice that for each $i$ we have that

$$\frac{d_i(x_i, y_i)}{i} \leq \frac{d_i(x_i, z_i)}{i} + \frac{d_i(z_i, y_i)}{i} \leq d(x,z) + d(z,y).$$

Taking the supremum over $i$ on the left yields the desired inequality.

Now we must prove that $d$ induces the product topology. Let $\tau_\Pi$ be the product topology and $\tau_d$ the metric topology.

For any $N \in \mathbb{N}$,

$$d(x,y) \leq \max \left\{ \frac{d_1(x_1, y_1)}{1}, \ldots, \frac{d_N(x_N, y_N)}{N}, \frac{1}{N} \right\}.$$

So given $\epsilon > 0$ and $x \in \prod M_i$, we can take $N > 1/\epsilon$ and

$$V_{x,\epsilon} = \left[ \prod_{i=1}^{N} B_{d_i}(x_i, \epsilon) \right] \times \left[ \prod_{i > N} M_i \right] \in \tau_\Pi.$$

Then for $y \in V_{x,\epsilon}$ we have

$$d(x,y) \leq \max \left\{ \frac{d_1(x_1, y_1)}{1}, \ldots, \frac{d_N(x_N, y_N)}{N}, \frac{1}{N} \right\} < \epsilon.$$

So that $V_{x,\epsilon} \subseteq B_d(x, \epsilon)$. This means that $\tau_d \subseteq \tau_\Pi$.

On the other hand, if $U = \prod U_i \in \tau_\Pi$ with $U_{\alpha_1}, \ldots, U_{\alpha_n}$ being the only open sets not equal to $M_i$, we can take $\epsilon_{\alpha_i} > 0$ such that $B_{d_{\alpha_i}}(x_{\alpha_i}, \epsilon_{\alpha_i}) \subseteq U_{\alpha_i}$. Then take $0 < \epsilon < \min\{\epsilon_{\alpha_i}/\alpha_i\}$.

For any $y \in B_d(x, \epsilon)$ and every $\alpha_i$,

$$\frac{d_{\alpha_i}(x_{\alpha_i}, y_{\alpha_i})}{\alpha_i} \leq d(x,y) \leq \epsilon < \frac{\epsilon_{\alpha_i}}{\alpha_i}.$$

Hence $y_{\alpha_i} \in B_{d_{\alpha_i}}(x_{\alpha_i}, \epsilon_{\alpha_i})$ and $y \in U$, which means that $B_d(x, \epsilon) \subseteq U$. That is, $\tau_\Pi \subseteq \tau_d$.

Therefore $d$ induces the product topology on $\prod M_i$, which makes this a sequentially compact metric space. ■