Problem Set 7 Problem 21

Let Y be compact Hausdorff, X arbitrary. Then $\pi : X \times Y \to X$ is a closed map.

Proof. Let $F \subset X \times Y$ be closed. We want to show that $\pi(F) \subset X$ is closed. If $\pi(F) = X$, we're done. If not, let $x \in \pi(F)^c$. Then we know that $(\{x\} \times Y) \cap F = \emptyset$, meaning F^c is an open set containing $\{x\} \times Y$. Since Y is compact, Hausdorff, we can apply the tube lemma: there exists open neighborhood $W_x \subset X$ of x such that $(W_x \times Y) \subset F^c$. But that implies $(W_x \times Y) \cap F = \emptyset$. So $W_x \cap \pi(F) = \emptyset$, meaning $W_x \subset \pi(F)^c$. This implies

$$\pi(F)^c = \bigcup_{x \in \pi(F)^c} W_x,$$

□

Problem Set 7 Problem 22

Let $f : X \to Y$ be a map (X a space, Y compact Hausdorff).

(i) If the graph $\Gamma_f \subset X \times Y$ is closed, f is continuous.

Proof. Suppose Γ_f is closed in $X \times Y$. Let $C \subset Y$ be closed. Then $X \times C$ is closed in $X \times Y$ so that $\Gamma_f \cap (X \times C)$ is closed. Since Y is compact Hausdorff, by Problem 21, the projection $\pi : X \times Y \to X$ is a closed map. So $\pi(\Gamma_f \cap (X \times C))$ is closed. Observe that $\pi(\Gamma_f \cap (X \times C)) = f^{-1}(C)$. Indeed, if $\Gamma_f \cap (X \times C) = \emptyset$, then $f(x) \notin C$ for each $x \in X$. Thus, $f^{-1}(C) = \emptyset = \Gamma_f \cap (X \times C)$. If $\Gamma_f \cap (X \times C) \neq \emptyset$,

$$x \in \Gamma_f \cap (X \times C) \iff f(x) \in C \iff x \in f^{-1}(C).$$

Thus, $f^{-1}(C)$ is closed. Since C was arbitary closed set in Y, f^{-1} takes closed sets to closed sets, which is equivalent to saying f is continuous. □

(ii) If f is continuous, Γ_f is closed.

Proof. Let $(x, y) \in \Gamma_f^c$. Then we have $f(x) \neq y$. Since Y is Hausdorff, there exists disjoint sets open U_x, V_y that are neighborhood of $f(x)$ and y respectively. f is continuous, so $f^{-1}(U_x)$ is open in X. Meaning $f^{-1}(U_x) \times V_y$ is open in $X \times Y$. Since $f(x) \in U_x$, we know $x \in f^{-1}(U_x)$. So $(x, y) \in f^{-1}(U_x) \times V_y$.

Now, observe that

$$\Gamma_f \cap (f^{-1}(U_x) \times Y) \subset (f^{-1}(U_x) \times U_x).$$

Thus,

$$\begin{aligned}\Gamma_f \cap (f^{-1}(U_x) \times V_y) &= (\Gamma_f \cap (f^{-1}(U_x) \times Y)) \cap (f^{-1}(U_x) \times V_y) \\ &\subset (f^{-1}(U_x) \times U_x) \cap (f^{-1}(U_x) \times V_y) \\ &= \emptyset,\end{aligned}$$

since U_x and V_y are disjoint. Thus, $\Gamma_f \cap (f^{-1}(U_x) \times V_y) = \emptyset$. Therefore, we have $(x, y) \in (f^{-1}(U_x) \times V_y) \subset \Gamma_f^c$ for each $(x, y) \in \Gamma_f^c$. This implies

$$\Gamma_f^c = \bigcup_{(x,y)\in\Gamma_f^c} (f^{-1}(U_x) \times V_y),$$

which implies Γ_f^c is open since each $f^{-1}(U_x) \times V_y$ is open. Thus, Γ_f is closed. □