Topology Day 2

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Let (X, τ) be a topological space and $E \subseteq X$. Define

$$\overline{E} := \bigcap \{F \mid E \subseteq F, F \text{ closed}\}$$

to be the *closure* E.

Define $x \in E$ to be a *limit point* of E if $\forall U \in \tau$ with $x \in U$, $U \cap E$ contains a point other than x.

Denote $E' := \{x \in X \mid x \text{ is a limit point of } E\}$.

We prove that $\overline{E} = E \cup E'$.

First we show that $\overline{E} \subseteq E \cup E'$. Let $x \in \overline{E}$. By definition,

$$x \in \bigcap_{E \subseteq F \text{ closed}} F,$$

so $\forall F \supseteq E$, $x \in F$. If $x \in E$, $x \in E \cup E'$, so we are done. Otherwise, if $x \notin E$, then we must show that $x \in E'$. That is, we must show that $\forall U \in \tau$ with $x \in U$, $U \cap E$ contains a point other than x. Since we assume that $x \notin E$, this is the same as showing $U \cap E \neq \emptyset$.

Instead, assume to the contrary that $E \cap U = \emptyset$. Then $E \subseteq U^c$, which is closed, so $x \in U^c$. But we assume that $x \in U$, so we have a contradiction.

Next we show that $E \cup E' \subseteq \overline{E}$. Let $x \in E \cup E'$. Then either $x \in E$ or $x \in E'$. If $x \in E$, then $x \in E \subseteq F$ closed, so $x \in F$ $\forall F \supseteq E$ closed. Hence

$$x \in \bigcap_{E \subseteq F \text{ closed}} F = \overline{E}.$$

Otherwise, if $x \in E'$, then $\forall U \in \tau$ with $x \in U$, $U \cap E$ contains a point other than x. Using this, we must show that

$$x \in \bigcap_{E \subseteq F \text{ closed}} F,$$

which is the same as showing that

$$x \notin \bigcup_{E \subseteq F \text{ closed}} F^c.$$

Suppose for the sake of contradiction that $x \in F^c$ for some $F \supseteq E$ closed. Then $F^c \subseteq E^c$, and since F^c is open, $\exists U \in \tau$ with $x \in U \subseteq F^c \subseteq E^c$. But then $U \cap E = \emptyset$, a contradiction as $x \in E'$. $\square$