TOPOLOGY HW 1

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Problem 8.7) First, it will be convenient to appeal to the fairly standard notation

$$B_r(x,y) := \left\{(u,v) \in \mathbb{R}^2 \mid (x-u)^2 + (y-v)^2 < r^2\right\}$$

for the Euclidean metric balls in the plane $\mathbb{R}^2$. Define $U = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ and $L = \{(x,0) \in \mathbb{R}^2\}$. We define a topology on the half-plane $U \cup L$ as follows: define the set $\mathcal{B}_1 = \{B_r(x,y) \mid 0 < r < y\}$ of balls contained in U and $\mathcal{B}_2 = \{B_\delta(x,\delta) \cup (x,0)\}$ of balls tangent to L, including the single point at which it is tangent to L. The collection $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for a topology on $U \cup L$, producing a topological space called H, the Moore Half-Plane.

(i) We show $L \subset H$ inherits the discrete topology as a subspace of H. Let A be any subset of L, and define $\mathcal{A} = \{B_1(a,1) \cup \{a,0\} : (a,0) \in A\} \subset \mathcal{B}$. This is collection of open sets in H, so $\bigcup_{B \in \mathcal{A}} B$ is an open set which intersects L at exactly A, so A is open in the subspace topology on L. Since A was arbitrary, any subset of L is open, so this subspace is discrete.

(ii) The collection $\mathcal{B}_1$ of basis elements covers U without intersecting L, so $\bigcup_{B \in \mathcal{B}_1} B = U$ is open—implying its complement L is closed in H.

(iii) The set $\mathbb{Q}^2 \cap U$ of rational points above L intersects every $B \in \mathcal{B}$, since each such B contains an Euclidean ball in the upper half-plane U and the rational points are known to intersect every such ball in the plane. The rational points are also countable, so $\mathbb{Q}^2 \cap U$ is a countable dense subspace of H, so the Moore half-plane is separable. However, since L is uncountable and inherits the discrete topology, then L is *not* separable.