COVERING DIMENSION AND EMBEDDING THEOREMS

The following theorem generalizes the basic upper bound for finite unions of closed subspaces to the case of countably many compact subspaces exhausting a space. The outline of proof (with details left as exercises) is an expanded version of [Munkres 1, p.316] (see also [Munkres 2], p. 23-24).

Theorem 1. Let X be a σ-compact Hausdorff space. If every compact subspace of X has covering dimension $\leq m$, then X has covering dimension $\leq m$.

(Recall σ-compact means $X = \bigcup_{n\geq 1} X_n$, with X_n compact and $X_n \subset int(X_{n+1})$.)

Proof. We must show that, given an open cover $\mathcal{A}$ of X, there exists a second open cover $\mathcal{B}$ refining $\mathcal{A}$, with order $\leq m+1$.

Exercise 1. Show there exists an open cover $\mathcal{B}_0$ refining $\mathcal{A}$, with the property that if $B \in \mathcal{B}_0$ intersects X_n, then $B \subset X_{n+1}$.(Of course this property is inherited by any cover refining $\mathcal{B}_0$.)

Hint: Consider the collection of sets $\{U \cap (int(X_{n+1}) \setminus X_{n-1}); U \in \mathcal{A}\}$ for all n (with a technical correction for low n.)

The idea to prove the theorem is to construct a sequence of successive refinements of $\mathcal{B}_0$ (hence of $\mathcal{A}$):

$$\mathcal{B}_0 < \mathcal{B}_1 < \ldots < \mathcal{B}_n < \mathcal{B}_{n+1} < \ldots,$$

where each $\mathcal{B}_n$ has order $\leq m+1$ over X_n. Then define the open cover $\mathcal{B}$ as:

$$\mathcal{B} = \{B \in \bigcup_n \mathcal{B}_n; (\exists N \geq 1)(\forall n \geq N) B \in \mathcal{B}_n\};$$

we might say $\mathcal{B} = \liminf_n \mathcal{B}_n$: the collection of sets B which are eventually in each $\mathcal{B}_n$.

Exercise 2. Show that, assuming $\mathcal{B}$ covers X, it has order $\leq m+1$ (clearly $\mathcal{B}$ refines $\mathcal{A}$).

From prior results, we know that for each n there exists an open cover $\mathcal{C}_n$ of X refining $\mathcal{B}_0$ which has order $\leq m+1$ at points of X_n, and we can take these covers to be successively refining. Why not just take the $\mathcal{C}_n$ as the $\mathcal{B}_n$? The reason is we don't have enough control to guarantee there are enough sets in $\liminf_n \mathcal{C}_n$ to cover X. Instead, assuming (inductively) $\mathcal{B}_n$ is given (a cover of X with order $\leq m+1$ at points of X_n), let $\mathcal{C}_{n+1}$ be an

open cover of X which refines $\mathcal{B}_n$ and has order $\leq m+1$ at points of X_{n+1}. Then define $\mathcal{B}_{n+1}$ to consist of the following open sets:
(Type 1): If $B \in \mathcal{B}_n$ and B intersects X_{n-1}, add B to $\mathcal{B}_{n+1}$. (Note that in this case $B \subset X_n$, by Exercise 1.)
(Type 2): Let $f : \mathcal{C}_{n+1} \to \mathcal{B}_n$ be a 'choice function' for this refinement of covers (so $C \subset f(C)$.) Then for $B \in \mathcal{B}_n$, consider the open subset of B:

$$D_B := \bigcup \{C \in \mathcal{C}_{n+1}; f(C) = B\}$$

(the old preimage trick.) If $B \in \mathcal{B}_n$ and $B \cap X_{n-1} = \emptyset$, add D_B to the collection $\mathcal{B}_{n+1}$. (Note $D_B \subset B$, so in this case we also have $D_B \cap X_{n-1} = \emptyset$.)

And now define $\mathcal{B} = \liminf_n \mathcal{B}_n$, as defined above (the collection of B which are in all $\mathcal{B}_n$ for n sufficiently large).

We have three things to prove:

Exercise 3. (i) (Easy) $\mathcal{B}$ refines each $\mathcal{B}_n$, and therefore refines $\mathcal{A}$.

(ii) $\mathcal{B}$ covers X.

Hint: First show that if $B \in \mathcal{B}_n$ and $B \cap X_{n-1} \neq \emptyset$, then $B \in \mathcal{B}_m$ for all $m \geq n$, so $B \in \mathcal{B}$.

Then conclude the proof of (ii) as follows:

Given $x \in X$, say $x \in X_n$ and $x \in B \in \mathcal{B}_n$. If $B \cap X_{n-1} \neq \emptyset$, we just saw $B \in \mathcal{B}$. Otherwise, let $x \in C \in \mathcal{C}_{n+1}$, and let $B' = f(C) \in \mathcal{B}_n$, so $x \in B'$. If $B' \cap X_{n-1} \neq \emptyset$, then $B' \in \mathcal{B}$. If $B' \cap X_{n-1} = \emptyset$, then we have $x \in D_{B'} \in \mathcal{B}_{n+1}$. Show that we have: $D_{B'} \in \mathcal{B}$.

Thus in any case x lies in a set of the limit family $\mathcal{B}$.

(iii) Show that $\mathcal{B}_{n+1}$ has order $\leq m+1$ at points of X_{n+1}.

Hint: Let $x \in X_{n+1}$, and suppose $x \in U_1 \cap \ldots \cap U_k$, the intersection of k sets in $\mathcal{B}_{n+1}$, none of which is contained in any of the others. We need to show $k \leq m+1$. Each U_i is either Type 1 or Type 2. If it is of type 1, $U_i \in \mathcal{B}_n$ and $U_i \subset X_n$. (See Exercise 1.)

Consider two cases: $x \in X_n$ and $x \in X_{n+1} \setminus X_n$. In the second case, explain why all the U_i must be of Type 2, and why then $k \leq m+1$.

In the first case ($x \in X_n$) we have $U_i \in \mathcal{B}_n$ (if of Type 1) or $U_i = D_{B_i} \subset B_i \in \mathcal{B}_n$, and a U_j of the first kind cannot coincide with a B_i of the second kind if $i \neq j$, since none of the U_i is contained in any other. Thus $x \in X_n$ is in the intersection of k distinct sets in $\mathcal{B}_n$, so $k \leq m+1$ (since $\mathcal{B}_n$ is assumed to have order $\leq m+1$ at points of X_n.) This concludes the proof of Theorem 1!

Exercise 4. Explain how the following corollaries follow from the theorem: (i) Any topological m-manifold (compact or not) has covering dimension at most m. (ii) Every closed subspace of R^N has covering dimension at most N.

Combining the upper bound for closed subspaces of euclidean space with the embedding theorem to be proved below, we have:

Corollary. A space X admits a continuous proper embedding into R^N (for some N) if and only if X is locally compact, Hausdorff with countable basis and has uniformly bounded covering dimension on compact sets (that is, there exists an m so that for all $C \subset X$ compact, $covdim(C) \leq m$.)

Exercise 5: Prove this corollary, assuming the embedding theorem below (Theorem 2).

The next result is an embedding theorem for σ-compact spaces with bounded covering dimension on compact subspaces, generalizing the result for compact metric spaces of finite covering dimension. The proof outlined below (with details left as exercises) is adapted from [Munkres 1, p.315].

Theorem 2. Let X be a σ-compact, second countable Hausdorff space such that each compact subspace has covering dimension $\leq m$ (for some fixed m independent of the subspace.) Then there exists a continuous, proper embedding $f : X \to R^{2m+1}$.

In the space $C(X, R^N)$ we take the uniform topology. To define a compatible metric, let $\bar{d}$ be a bounded metric in R^N, for example:

$$\bar{d}(v, w) = \frac{||v - w||}{1 + ||v - w||},$$

then set $\rho(f, g) = \sup\{\bar{d}(f(x), g(x)); x \in X\}$.

Recall that a map $f : X \to R^N$ is *proper* if the preimage of a compact set is compact in X. Equivalently, if $x_i \to \infty$ in X implies $f(x_i) \to \infty$ in R^n. (Where, by definition, $x_i \to \infty$ in X means for any $C \subset X$ compacf, there exists $i_0 \geq 1$ so that $x_i \notin C$ if $i \geq i_0$.)

Exercise 6. (i) Prove there exists a continuous, proper function from X to R (if X is σ-compact, second countable Hausdorff.) Of course, this implies there exist proper maps from X to R^N, for any N.

Hint: Consider the compact exhaustion $X = \bigcup_n C_n$, C_n compact, $C_n \subset int(C_{n+1})$, and the open cover $\{U_n = int(C_n)\}_{n\geq 1}$ of X. Let $(\phi_n)_{n\geq 1}$ be a continuous partition of unity strictly subordinate to this cover, and consider the function:

$$f : X \to R, \quad f(x) = \sum_{n=1}^{\infty} n\phi_n(x).$$

(ii) Prove that if $f \in C(X, R^N)$ is a proper map, $g \in C(X, R^N)$ and $\rho(f, g) < 1$, then g is a proper map as well.

Remark: If X is a smooth manifold and $\{\phi_n\}_{n\geq 1}$ is a smooth partition of unity, the function f will be smooth and proper: a 'smooth exhaustion function' for f.

Given $C \subset X$ compact and $\epsilon > 0$, define:

$$U(\epsilon, C) = \{f \in C(X, R^N); \Delta(f_{|C}) < \epsilon\},$$

where recall:

$$\Delta(f_{|C}) = \sup\{diam_X(f^{-1}(y) \cap C); y \in R^N\}.$$

Exercise 7. Show that $U(\epsilon, C)$ is open in $C(X, R^N)$ (with the uniform topology.)

Exercise 8. Assume $N = 2m + 1$. Show that for all $\epsilon > 0$ and all $C \subset X$ compact, $U(\epsilon, C)$ is dense in $C(X, R^N)$.

Hint. Given $f \in C(X; R^N)$, $C \subset X$ compact and $\epsilon > 0, \delta > 0$, use a prior result to find $g : C \to R^N$ continuous with $\Delta(g) < \epsilon$ and $\bar{d}(f(x), g(x)) < \delta$ for $x \in C$. Then extend g to $h : X \to R^N$ (continuous and δ-close to f in the metric ρ) by applying the Tietze extension theorem to $f - g$.

By Baire's theorem, it follows from the results of these two exercises that the set $\bigcap_{n\geq 1} U(\frac{1}{n}, C_n)$ is dense in $C(X, R^N)$ (in the uniform topology), in particular non-empty.

Exercise 9. Show that if $g \in \bigcap_{n\geq 1} U(\frac{1}{n}, C_n)$, then g is injective on X.

Exercise 10. Put everything together to write a proof of Theorem 2.

An interesting application of covering dimension is the (perhaps surprising) fact that any topological m-manifold, compact or not, may be covered by $m + 1$ domains of local charts (in general not connected.) To see this, start with the lemma:

Lemma. Let M be a topological m-manifold. (More generally, a second-countable paracompact Hausdorff space of covering dimension $\leq m$.) Let $\mathcal{A}$

be an open cover of M. Then there exists a locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$ which is the union of $m+1$ collections of open sets $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m$, where each $\mathcal{B}_i$ consists of *disjoint* open sets.

Remark. (i) It is clear that, conversely, if a space has the property that any open cover admits a refinement of this form, its covering dimension is $\leq m$.

(ii) For the case $M = R^m$, a geometric proof is given in [Munkres 1] (p. 306, 313).

Proof. (outline; cp. [Munkres 2], p.20 for the details.) Since M is a Lindelöf space, we may assume $\mathcal{A} = (U_k)_{k\geq 1}$ is countable. Let $(V_j)_{j\geq 1}$ be an open refinement of order $\leq m+1$, $V_j \subset U_{k(j)}$ for each $j \geq 1$ (for some choice function $k(j)$). Let (ϕ_j) be a continuous partition of unity strictly subordinate to (V_j) (which always exists on paracompact Hausdorff spaces, [Munkres 1, p.259]).

Given $i_0 \geq 1$, let:

$$W(i_0) = \{x \in M; (\forall i \neq i_0)\phi_i(x) < \phi_{i_0}(x)\}.$$

And then let $\mathcal{B}_0$ be the collection of all such $W(i_0)$ (clearly all disjoint and open), for varying $i_0 \in \mathbb{N}$.

Now given two indices $i_0 < i_1$, set:

$$W(i_0, i_1) = \{x \in M; (\forall i \notin \{i_0, i_1\})\phi_i(x) < \min\{\phi_{i_0}(x), \phi_{i_1}(x)\}.$$

Varying i_0, i_1 we get a collection of disjoint open sets, and call that $\mathcal{B}_1$.

And so on until, considering all possible collections of multi-indices of length $m-1$, we define the collection $\mathcal{B}_m$ of disjoint open sets. (*Question:* What happens if we try to continue past this multi-index length?)

To see the union of these collections $\mathcal{B}_0, \ldots, \mathcal{B}_m$ covers M, let $x \in M$ and consider the finite collection F of indices k such that $\phi_k(x) > 0$. F has cardinality n_0 at least 1, and at most $m+1$ (since (V_j) has order $\leq m+1$). And for each $i \notin F$, $\phi_i(x) = 0$; so $x \in W(F) \in \mathcal{B}_{n_0-1}$.

(For the proof of local finiteness of the union of the $\mathcal{B}_i$, see [Munkres 2], p.21.)

Here is the promised application of the lemma:

Proposition. Let M be a topological m-manifold (compact or not). Then M may be covered by $m+1$ domains of local charts $\psi_i : U_i \to R^m$,

$i = 0, \dots m$. In fact we may choose the U_i so that each $U_i = \bigsqcup_j V_{ij}$ (disjoint union of countably many open sets), and so that the $\psi_i(V_{ij})$ are bounded open sets in R^m, disjoint for varying j, with only finitely many inside any closed ball at the origin $B_R \subset R^m$.

Proof. This is problem 2.8 in [Munkres 2]. Think about it for a bit.

One application of this result is extending the proof of existence of smooth embeddings into some euclidean space from compact to non-compact manifolds; namely, the following smooth embedding theorem:

Theorem 3. Let M be a smooth non-compact m-dimensional manifold. (Or C^r, $r \geq 1$.) There exists a smooth proper embedding $F : M \to R^{(m+1)^2}$.

Proof. This is similar to the compact case. Let $\{U_i, \psi_i\}_{i=0}^m$ be a smooth atlas for M as given in the proposition. Let $\{\phi_i\}_{i=0}^m$ be a smooth partition of unity strictly subordinate to the finite cover $\{U_i\}_{i=0}^m$. Consider the smooth map from M to $R^{(m+1)^2}$:

$$F(x) = (\phi_0(x), \dots \phi_m(x), \phi_1(x)\psi_1(x), \dots, \phi_m(x)\psi_m(x)),$$

where we extend $\phi_i\psi_i$ as $0 \in R^m$ on $M \setminus U_i$. One shows F is an injective immersion exactly as in the compact case.

Claim. F is a proper map. One way to see this is to consider a sequence $x_l \in M$ such that $x_l \to \infty$ and prove that $F(x_l) \to \infty$ in $R^{(m+1)^2}$.

Proceeding by contradiction, suppose instead that for infinitely many sequence elements x_l, their images $F(x_l)$ lie entirely within some closed ball at the origin $B_R \subset R^m$. Denote the corresponding subsequence still by x_l, so $x_l \to \infty$ on M.

Since for each $l \geq 1$, $\phi_0(x_l) + \ldots + \phi_m(x_l) = 1$, it is easy to see that $\lim_{l\to\infty} \phi_i(x_l)$ cannot be zero for all i. So for some $0 \leq i_0 \leq m$ we must have $\limsup_l \phi_{i_0}(x_l) = c > 0$. Taking a further subsequence, we may assume $\phi_{i_0}(x_l) \geq c/2$, for all l sufficiently large ($l \geq L$). Necessarily then, for $l \geq L$ the x_l must lie in $U_{i_0} = \bigsqcup_{j\geq 1} V_{i_0 j}$. Each x_l is in exactly one set $V_{i_0 j}$, at most finitely many are in any given $V_{i_0 j}$ (since these sets are precompact and $x_l \to \infty$) and by construction only finitely images $\psi_{i_0}(V_{i_0 j})$ are found in each closed ball at the origin $B_r \subset R^m$. Thus we must have $\psi_{i_0}(x_l) \to \infty$ in R^m, and since $\phi_{i_0}(x_l) \geq c/2$ for $l \geq L$, we see that $\phi_{i_0}(x_l)\psi_{i_0}(x_l) \to \infty$ in R^m. This is a component of $F(x_l)$, so $F(x_l) \to \infty$ in R^m, contradiction.

Another approach. It is possible to prove the existence of an injective immersion of a smooth non-compact m-manifold to R^{4m+3}, without appeal to covering dimension. Here is the argument. Let $f : X \to R_+$ be a smooth exhaustion function (i.e. proper) and for each $n \geq 0$ let $X_n = f^{-1}([n, n+1])$. Cover X_n by finitely many coordinate charts $U_1, \ldots, U_k$. Now let $Y_n = (U_1 \cup \ldots \cup U_k) \cap f^{-1}((n - 1/3, n + 1/3))$. Then each Y_n is an open submanifold of X, with $X_n \subset Y_n$ and $Y_n \cap Y_p = \emptyset$ if $|n - p| \geq 2$. Each Y_n can be covered by finitely many coordinate charts, so there exists an injective immersion from Y_n to some R^K, and by the projection argument an injective immersion $\psi_n : Y_n \to R^{2m+1}$.

Let ϕ_n be a smooth function on X with support contained in X_n, and identically 1 in an open neighborhood of X_n. (*Exercise:* prove that such a function exists.) Define:

$$F : X \to R^{4m+3}, \quad F(x) = (\sum_{n \in odd} \phi_n(x)\psi_n(x), \sum_{n \in even} \phi_n(x)\psi_n(x), f(x)).$$

(Note at most one term in each of the sums is nonzero, for any given $x \in X$, so F is a smooth map.) *Claim:* F is an injective immersion.

To see that F is injective, let $F(x) = F(y)$, so $f(x) = f(y) \in [n, n+1]$ for some n. Thus $x, y \in X_n \subset Y_n$, and $\psi_n(x) = \psi_n(y)$. Since ψ_n is injective, $x = y$.

To see that F is an immersion, just note that if $x \in X_n$ and $v \in T_x X$, one component of $dF(x)[v]$ is $d\psi_n(x)[v]$, and ψ_n is an immersion.

Remark. It is tempting here to try to use the same idea used in the compact case (successive orthogonal projection on hyperplanes) to reduce the codimension of the embedding. The problem is that orthogonal projection p_H (from R^N to a hyperplane H) is not a proper map, so in general neither would be $p_H \circ F$. A different idea is needed to prove the following theorem.

Theorem 4. Let M^m be a smooth non-compact manifold. There exists a proper injective immersion from M to R^{2m+1} (that is, a proper embedding.)

Proof. Let $f : M \to R^{2m+1}$ be an injective immersion. (Obtained from the map in Theorem 3 by the Whitney projection technique.) By composing with the diffeomorphism $y \mapsto \frac{y}{1+|y|}$ from R^{2m+1} to the unit ball, we may assume $|f(x)| < 1$ for all $x \in M$. Let $\phi : M \to R_+$ be a smooth exhaustion function (i.e. proper.) Consider the map:

$$\tilde{f} : M \to R^{2m+1} \times R, \quad \tilde{f}(x) = (f(x), \phi(x)).$$

Clearly $\tilde{f}$ is an injective immersion (since f is.) By the Whitney argument, we know that the composition of $\tilde{f}$ with projection onto the hyperplane $H_v = \langle v \rangle^{\perp}$ is still a projective immersion, for a dense set of $v \in S^{2m+1}$.

We *claim* that if $v = (v', w) \in R^{2m+1} \times R$ with $v' \neq 0$ (or $|w| < 1$), this projection $\pi_v \circ \tilde{f} : M \to H_v$ is also proper. Indeed, for $(y', y) \in R^{2m+1} \times R$,

$$\pi_v(y', y) = (y', y) - [y' \cdot v' + yw](v', w).$$

So for $g = \pi_v \circ \tilde{f}$, we have:

$$g(x) = (z', \phi(x) - [f(x) \cdot v' + \phi(x)w]w) \in R^{2m+1} \times R, \qquad \text{(for some } z' \in R^{2m+1}\text{)}.$$

Let $K \subset H_v$ be a compact set. Then there exists an $A > 0$ so that $|y| \leq A$, whenever $(y', y) \in K$. Thus. if $x \in g^{-1}(K)$, we have
$|\phi(x) - [f(x) \cdot v' + \phi(x)w]w| \leq A$, and hence:

$$(1 - w^2)|\phi(x)| \leq A + |f(x)| \leq A + 1.$$

This implies $g^{-1}(K) \subset L$, where $L \subset M$ is the compact set:

$$L = \phi^{-1}[-\frac{A+1}{1-w^2}, \frac{A+1}{1-w^2}],$$

proving the claim.