MATH 561, FALL 2022–PROBLEM SET 3 (due Thursday 9/29)

1. M, N, P differentiable manifolds, $f : M \to N$ C^k surjective submersion, $g : N \to P$ continuous. Prove that if $g \circ f : M \to P$ is C^k, then g is C^k.

2. Let $f : M \to N$ be a C^k map, transversal to a submanifold $S \subset N$. Let $V \subset M$ be the submanifold $V = f^{-1}(S)$, assumed non-empty. Then if $p \in V$, prove the tangent space T_pV is:

$$T_pV = (df(p))^{-1}[T_{f(p)}S],$$

the preimage under the differential of f.

3. (a) If X is compact and Y connected, every submersion $f : X \to Y$ is surjective.

(b) There exist no submersions of compact manifolds into Euclidean spaces.

4. Let p be a real polynomial function in k variables, homogeneous of degree m. Prove that the levels sets $\{p(x) = a\}$ are $k-1$-dimensional submanifolds of R^k, provided $a \neq 0$. Show that the manifolds with $a > 0$ are all diffeomorphic to each other, and same for those with $a < 0$.

5. (a) Prove that the group $SL(n)$ ($n \times n$ matrices with real entries and determinant 1) is a submanifold of $M(n)$ (vector space of $n \times n$ real matrices), and thus a Lie group. *Hint:* prove that 0 is the only critical value of the determinant function.

(b) Describe (with justification) the tangent space to $SL(n)$ at the identity.

6 Prove that the set of real 2×2 matrices of rank 1 is a three-dimensional submanifold of $M(2) = R^4$.

7. (a) Let V be a finite-dimensional real vector space, $\Delta \subset V \times V$ the diagonal. For a linear map $A \in \mathcal{L}(V)$, show that the graph $W = \{(v, Av); v \in V\} \subset V \times V$ is transversal to the diagonal if and only if 1 is not an eigenvalue of A.

(b) *Def.* A smooth map $f : X \to X$ is a *Lefschetz map* if at any fixed point x of f (that is, $f(x) = x$), 1 is not an eigenvalue of the differential $df(x) \in \mathcal{L}(T_xX)$.

Prove that a Lefschetz map of a compact manifold has only finitely many fixed points.