Topology HW 5

Elisha Brooks

April 26, 2021

3 Show that any (smooth) map $f : S^n \to S^n$ with degree different from $(-1)^{n+1}$ must have a fixed point.

This is most easily shown by instead showing the contrapositive: If f has no fixed points, then it must have degree $(-1)^{n+1}$. In other words, f must be homotopic to the antipodal map on S^n. So we define a homotopy in S^n given by

$$F(t,x) = \frac{(1-t)f(x) - tx}{||(1-t)f(x) - tx||}$$

To make sure this is well-defined, we ensure the denominator is never zero:

$$\begin{aligned} (1-t)f(x) - tx &= 0 \\ (1-t)f(x) &= tx \\ \implies (1-t)||f(x) &= t||x|| \end{aligned}$$

Since both $f(x)$ and x have norm 1

$$\begin{aligned} 1 - t &= t \\ t &= 1/2 \end{aligned}$$

But observe that at $t = 1/2$:

$$\begin{aligned} (1 - \frac{1}{2})f(x) - \frac{1}{2}x &= 0 \\ f(x) &= x \end{aligned}$$

which is a contradiction. Hence F is a valid homotopy. Therefore since f is homotopic to the antipodal map, it must have degree $(-1)^{n+1}$, by theorem B in Milnor.