MATH 562 - PROBLEM SET 5

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4 (i). Let $f : U \to \mathbb{R}^k$ be any smooth map defined on an open subset U of $\mathbb{R}^k$, and let x be a regular point, with $f(x) = z$. Let B be a sufficiently small closed ball centered at x, and define $\partial f : \partial B \to \mathbb{R}^k$ to be the restriction of f to the boundary of B. Prove that $W(\partial f, z) = +1$ if f preserves orientation at x and $W(\partial f, z) = -1$ if f reverses orientation at x.

Proof. By composing with translations, we may assume without loss of generality that $x = 0 = z$. Since 0 is a regular point, $df_0 : \mathbb{R}^k \to \mathbb{R}^k$ is a surjection, and thus an isomorphism. Then by the inverse function theorem, there exists a neighborhood V of 0 such that f carries V diffeomorphically onto a neighborhood of 0. Let B be a ball centered at 0 whose closure is contained in V. Let $A : \partial B \to \mathbb{R}^k$ be the restriction of df_0 to ∂B. Since $f(x) = df_0 + \varepsilon(x)$ for some function $\varepsilon(x)$ where $\varepsilon(x)/|x| \to 0$ as $|x| \to 0$, we may write $\partial f(x) = A(x) + \varepsilon(x)$. We wish to show $W(\partial f, 0) = W(A, 0)$. Consider the direction maps

$$\frac{\partial f(x)}{|\partial f(x)|} = \frac{A(x) + \varepsilon(x)}{|A(x) + \varepsilon(x)|} \quad \text{and} \quad \frac{A(x)}{|A(x)|},$$

and define $H : \partial B \times I \to S^{k-1}$ by

$$H(x,t) = \frac{A(x) + \varepsilon(tx)/t}{|A(x) + \varepsilon(tx)/t|} \quad \text{and setting} \quad H(x,0) = \frac{A(x)}{|A(x)|}.$$

Then H is continuous at $t = 0$ since $\varepsilon(tx)/t \to 0$ as $t \to 0$, and defines a homotopy between the two direction maps above. Since the degree of a map is a homotopy invariant,

$$\deg\Big(\frac{\partial f(x)}{|\partial f(x)|}\Big) = \deg\Big(\frac{A(x)}{|A(x)|}\Big),$$

i.e., $W(\partial f, 0) = W(A, 0)$. So now we must show that $W(A, 0) = \pm 1$, depending on whether f preserves or reverses orientation at 0. However A is the restriction of the linear isomorphism df_0. If f preserves orientation, then so does df_0, in which case df_0 is homotopic to the identity, implying A is homotopic to the identity map $\partial B \to \partial B$ and so $W(A, 0) = 1$. Similarly, if f reverses orientation, A is homotopic to the reflection map $(x_1, x_2, \ldots, x_k) \mapsto (-x_1, x_2, \ldots, x_k)$ and $W(A, 0) = -1$. □

4 (ii). Let $f : B \to \mathbb{R}^k$ be a smooth map defined on some closed ball B in $\mathbb{R}^k$. Suppose that z is a regular value of f that has no preimages on the boundary sphere ∂B, and consider $\partial f : \partial B \to \mathbb{R}^k$. Prove that the number of preimages of z, counted with our usual orientation convention, equals the winding number $W(\partial f, z)$.

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Proof. For each x_i in the preimage of z, by Problem 1, find a closed ball $B_i \subset B$ centered at x_i so that $W(f|_{\partial B_i}, z) = \pm 1$ depending on whether f preserves or reverses orientation at x_i. Then the number of points in the preimage of z, counted with our orientation convention, is

$$\sum_i W(f|_{\partial B_i}, z).$$

Let $B' = B - \bigcup_i B_i$. Then $\partial B' = \partial B \cup (-\bigcup_i \partial B_i)$, and

$$W(f|_{\partial B'}, z) = W(\partial f, z) - \sum_i W(f|_{\cup_i B_i}, z).$$

However, $W(f|_{\partial B'}, z) = 0$ since the directional map $\partial B' \to S^{k-1}$ extends to all of B'. Thus the number of points in the preimage of z, counted with our orientation convention, is $W(\partial f, z)$. □