Problem Set 5

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Problem 5. Let $B \subseteq \mathbb{R}^k$ be a closed ball and $f : \mathbb{R}^k \setminus \text{int}(B) \to Y$ be a smooth map defined outside of the open ball. Show that if the restriction ∂f of f to the boundary of the ball is homotopic to a constant, then f extends to a smooth map from $\mathbb{R}^k$ to Y.

Proof. To define f on the interior of B, we use the homotopy $F : \mathbb{R}^k \times I \to Y$, where $F(x, 0) = c \in Y$ and $F(x, 1) = \partial f(x)$. Without loss of generality, we assume that the ball B is centered at 0. Now define f on the interior of B by the equation

$$f(tx) = F(x, t)$$

where $x \in \partial B$ and $t \in [0, 1]$. f is clearly well-defined everywhere but 0, and at 0 it is also singular-valued since $F(x, 0) = c$ is constant. To ensure smoothness of the extension, we may substitute a suitable bump function into the definition of f for t, making t constant near 0 and constant near 1, and smooth throughout. □