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Math 562

Problem Set 5

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Pr. 6. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map with 0 as a regular value. Suppose that $f^{-1}(0)$ is finite and that the number of preimage points in $f^{-1}(0)$ is zero when counted with the usual orientation convention. Assuming the special case in dimension $k-1$, prove that there exists a mapping $g : \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = f$ outside a compact set.

Proof :

Since $f^{-1}(0)$ is finite, we can consider a closed ball B around the origin and $f^{-1}(0) \subset B$ with no points of $f^{-1}(0)$ on ∂B.
The number of preimage points in $f^{-1}(0)$ is zero, counted with the usual orientation convention, implies that the winding number $W(\partial f, 0)$ is zero by Pr.4(ii). Here $\partial f : \partial B \to \mathbb{R}^k \setminus \{0\}$ is the restriction of f.

Since ∂B is homotopic to S^{k-1}, we can define a smooth map $g' : \mathbb{R}^k \setminus \mathrm{Int}(B) \to \mathbb{R}^k \setminus \{0\}$ with $\partial g : S^{k-1} \to \mathbb{R}^k \setminus \{0\} = \partial f$ and $g' = f$ outside B.

By the corollary, $\partial g'$ is homotopic to a constant map.

Then by Pr.5, g' can be extended to a smooth map g defined on all of $\mathbb{R}^k$ into $\mathbb{R}^k \setminus \{0\}$.