Topology Homework

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9. (a) Let W be a compact $k+1$ dimensional manifold, with $f : \partial W \to S^k$ a smooth map. We already know that if f extends to a map of W into the sphere, then f has degree 0. Thus, all that remains to show is the converse: suppose that $\deg(f) = 0$. From problem 7, we know that this extends to a map $F : W \to \mathbb{R}^{k+1}$. Since degrees are invariant under homotopy, we may use the transversality homotopy lemma and the isotopy lemma, respectively, to assume that $0 \in \mathbb{R}^{k+1}$ is a regular value of F (so the set $F^{-1}(0)$ is a compact discrete space, or a finite collection of points) and so $F^{-1}(0) \subset B \subset U \subset \operatorname{Int} W$, where there exists a diffeomorphism $U \to \mathbb{R}^{k+1}$, and so we still have $\partial F = f$.

Consider the compact manifold with boundary $W' = W \setminus \operatorname{Int} B$, so that $\partial W' = \partial W - \partial B$ and if we let $G = \frac{F}{|F|}\Big|_{W'}$, then G is a well-defined map from W' to the sphere and $W(F|_{\partial W'}, 0)$ is consequently 0, since $F/|F|$ on the boundary $\partial W - \partial B$. But, we already know $W(F|_{\partial W}, 0) = W(f, 0) = 0$, and since $0 = \deg(G) = W(f, 0) - W(F|_{\partial B}, 0)$, then the winding number about 0 of F restricted to the boundary of B is also zero. Through the diffeomorphism $U \to \mathbb{R}^{k+1}$ and problem 5, the boundary map $F|_{U\setminus \operatorname{Int} B}$ extends to a smooth map $F_1 : U \to \mathbb{R}^{k+1} \setminus \{0\}$. But then, the map $H : W \to S^k$ defined by

$$H(z) = \begin{cases} \frac{F(z)}{|F(z)|} & \text{if } z \in W \setminus \operatorname{Int} B, \\ \frac{F_1(z)}{|F_1(z)|} & \text{if } z \in U, \end{cases}$$

is a well-defined smooth map, since F_1 and F agree on $U \setminus$ IntB, and extends the function f to a function from W into the sphere.

(b) If $f_0 : M \to S^k$ and $f_1 : M \to S^k$ are maps of the same degree, then if we define $W = I \times M$, where $I = [0, 1]$ is the compact unit interval, then W is a compact manifold of dimension $k + 1$ with boundary

$$\partial W = M_1 - M_0, \tag{1}$$

where $M_t = \{t\} \times M \subset W$ is given the orientation inherited as a natural copy of M inside W. Then, the boundary map $f : W \to S^k$ given by $f = f_0$ on M_0 and $f = f_1$ on M_1 is of degree 0, since (1) implies that $\deg(f) = \deg(f_1) - \deg(f_2) = 0$. But then, by the previous part of this problem, f extends to a smooth map $F : W \to S^k$, i.e., a homotopy between f_0 and f_1.